# Mathematics

## Grade 8

Mathematics Grade 8

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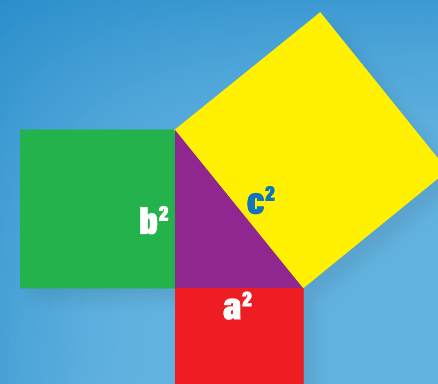


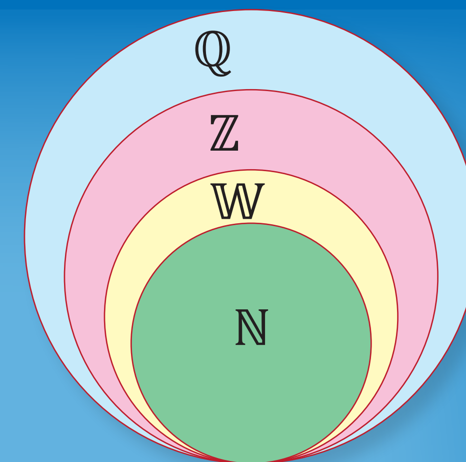


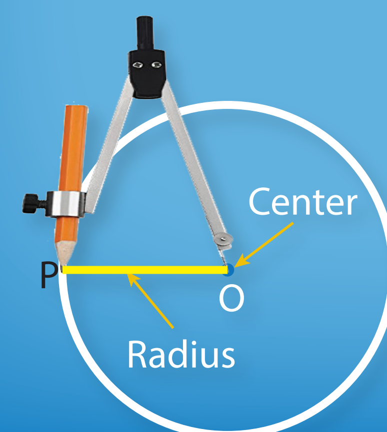


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## የጥንቃቄ መልዕክት

### ተማሪዎች!

### ይህን የተማሪ መጽሐፍ በጥንቃቄ ያዙ!

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# MATHEMATICS

## GRADE 8

## Student Textbook


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The textbook is prepared by Amhara National Regional State Education Bureau and the Scholars Council with budget allocated by the Amhara National Regional State Government.



The Scholars Council

## Rationales for the Curriculum Reform

Curriculum relevance and its appropriateness to develop higher order thinking skills have been a subject of discussion in Ethiopia for many years. Various studies have been conducted to identify and propose reform ideas to the general education quality and efficiency problems. The main and most comprehensive ones are the Education Roadmap study and the Cambridge Education study. These studies indicated that the general education curriculum is staffed with many subjects; some textbooks are overloaded with factual content; school contents are not adequately related with students' lives; and indigenous knowledge and real world problems are not integrated in the school curriculum. The studies also showed that the curriculum does not integrate ICT and has gaps to meet the needs of children with special educational needs. In addition, the studies demonstrated that curriculum materials do not thoroughly cultivate 21st Century skills and competencies such as lifelong learning, critical thinking, problem solving, creative and innovative thinking, communication and cooperative skills, leadership and decision making skills, technological skills, cultural identity and international citizenship. Consequently, these studies recommended the need to reform the curriculum.

Based on the recommendations, extended discussions and consultative meetings were conducted at national and regional levels with relevant stakeholders, teachers, parents, and educational leaders. Following these consultations, a new national curriculum framework, content flow chart, learning competencies, and syllabus have been prepared national level. Based on such documents, student textbooks and teacher guides of different subjects are prepared. The mathematics textbooks intend to engage students in the formulation and construction of mathematics knowledge and skill based on their day to day experiences and previous knowledge. Learners are expected to actively take part in drawing their prior knowledge and experience in the learning process.

## How to Use the Book

The book is prepared to enable student learn in active and participatory manner using their prior learning and experiences from the immediate environment. Students and teachers carry out activities and solve problems using their experiences and knowledge. In this process, students not only learn mathematical concepts and ideas but also develop the necessary learning to learn skills. Such practice also helps students to deeply understand the contents. To this end, teachers are expected to teach using the proposed learning and teaching strategies and learning processes. It is essential that students and teachers appreciate the processes involved in learning the contents and not merely focus on memorization of concepts and mathematical procedures. Hence, teachers are expected to employ the proposed techniques and implement all activities as they are designed by considering the objectives and contents of the textbook. In addition, teacher can select and use other methods and approaches based on students' capacity and needs.

## Dear Students!

You have to use the textbook with care. Learning largely determines the future of a generation. Learning is a base for any social, human, and economic development. The textbook contents and activities are designed to promote your active participation in class. By carrying out and studying all the activities, contents and questions provided in the textbook, you are required to develop deep understandings and skills. Effort, exercise, and perseverance are important to succeed in your academic career. You will enjoy and find learning mathematics to be fun! Make sure that you bring your textbook to class and use it during the teaching and learning process.

## Dear Parents!

Textbooks have significant roles to facilitate student learning. Thus, you are required to help and advice students to handle and use textbooks with care. Moreover, you are expected to motivate students to take textbooks to school and direct and support them to work the activities given by teachers. You should also visit your child's school to discuss with teachers about learning and behavioral change, identify gaps, and correct them through follow up and advising.

# Unit 1

# RATIONAL NUMBERS

## Learning outcomes:

**After completing this unit, you will be able to:**

- identify the relationships among the numbers systems: natural numbers ($\mathbb{N}$), whole numbers ($\mathbb{W}$), intergers ($\mathbb{Z}$) and rational numbers ($\mathbb{Q}$);
- define rational numbers;
- represent rational numbers on a number line;
- compare and order rational numbers;
- define absolute value of rational numbers;
- compute basic operations on rational numbers;
- appreciate application of rational numbers in real life problems.

## *Key terms*

- rational number
- comparing rational numbers
- operations on rational numbers
- terminating decimals
- simple interest
- compound interest
- opposite rational number
- ordering rational numbers
- absolute value
- repeating decimals

# Introduction

As human experience, needs and knowledge developed, the set of integers were not sufficient to describe the day-to-day activities and thus a new set of numbers, the set of rational numbers were developed. In your previous grades, you have learned about the sets of natural numbers, the set of whole numbers, the set of integers and fractions. In this unit, you will learn about the set of rational numbers, the four fundamental operations on rational numbers and some applications of rational numbers.

## 1.1 The Concept of Rational Numbers

In your previous grades you have learned about different number systems; the set of natural numbers, the set of whole numbers and the set of integers.

### Natural Numbers

Natural numbers, which are also called "counting numbers", are numbers that start with the number 1 and they do not involve negatives, fractions or decimals. You have started counting in your early ages with this set of numbers.

The set of natural numbers is denoted by $\mathbb{N}$ and it is given by: $\mathbb{N}=\{1,2,3,4,5,\cdots\}$.

### Whole Numbers

As the set of natural numbers was not enough to express the whole idea of counting, the idea of "zero" was developed. The set of natural numbers and zero "0" form a new class of numbers called the set of whole numbers, denoted by $\mathbb{W}$. That is, $\mathbb{W} = \{0, 1, 2, 3, 4, \cdots\}$ and observe that $\mathbb{N} \subseteq \mathbb{W}$.

### Integers

Whole numbers were not sufficient in order to represent situations like 'profit and loss', 'temperatures below 0°C, 'altitudes below sea level' etc. Thus, it was obvious, to extend the set of whole numbers by including negative numbers like $\cdots, -4, -3, -2, -1$ and the set of integers, denoted by $\mathbb{Z}$, was formed by combining the whole numbers and negative numbers. That is, $\mathbb{Z}=\{\cdots,-3,-2,-1,0,1,2,3,\cdots\}$

Observe that, you have the following relationships between the three given sets of numbers: $\mathbb{N} \subseteq \mathbb{W} \subseteq \mathbb{Z}$.

### Fractions

In our daily life there are quantities that cannot be expressed by the set of numbers that are mentioned above. For example, if you divide a bread into three equal parts, then each part cannot be expressed using integers. Thus, it was needed to introduce fractions. Fractions are numbers which can be written in the form $\frac{a}{b}$, where $a$ and $b$

are whole numbers and $b$ is not equal to zero ($b \neq 0$). In the fraction $\frac{a}{b}$, $a$ is called the numerator and $b$ is called the denominator.

## Operations on Fractions

Given two fractions $\frac{a}{b}$ and $\frac{c}{d}$, where $a, b, c$ *and* $d$ are natural numbers and $b \neq 0$ and $d \neq 0$:

1 $\frac{a}{b} + \frac{c}{d} = \frac{(a \times d) + (b \times c)}{b \times d}$ (Sum of fractions)

2 $\frac{a}{b} - \frac{c}{d} = \frac{(a \times d) - (b \times c)}{b \times d}$ (Difference of fractions)

3 $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$ (Product of fractions)

4 $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{a \times d}{b \times c}$ (Division of fractions and $c \neq 0$)

## Rational Numbers

### Definition 1.1

A number that can be written in the form $\frac{a}{b}$, where $a$ and $b$ are integers and $b \neq 0$ is called a rational number. The set of rational numbers, denoted by $\mathbb{Q}$, is defined by:

$\mathbb{Q} = \left\{\frac{a}{b} \middle| a, b \in \mathbb{Z} \text{ and } b \neq 0\right\}$.

### Example 1.1

Show that each one of the following numbers is a rational number.

$\frac{1}{2}, \frac{3}{2}, \frac{-5}{6}, 0, -1{,}5, 0.3$

**Solution**

i. $\frac{1}{2}$ is a rational number as it can writen in the form $\frac{a}{b}$ where $a = 1$ and $b = 2$.

ii. $\frac{3}{2}$ is a rational number as it can be written in the form $\frac{a}{b}$, where $a = 3$ and $b = 2$.

iii. $\frac{-5}{6}$ is a rational number as it can be written in the form $\frac{a}{b}$, where $a = -5$ and $b = 6$.

iv. $0 = \frac{0}{1}$, it is written in the form of $\frac{a}{b}$, where $a = 0$ and $b = 1$ are integers; so it is a rational number;

v. $-1.5 = \frac{-15}{10}$, it is written in the form of $\frac{a}{b}$, where $a = -15$ and $b = 10$; so it is a rational number;

vi. $0.3 = \frac{3}{10}$, it is written in the form of $\frac{a}{b}$, where $a = 3$ and $b = 10$; so it is a rational number;

## Note:

The rational number $\frac{a}{b}$ when $b \neq 0$, is called a fraction, $a$ is called the numerator and $b$ is called the denominator.

## Relationships between Integers and Rational Numbers

In your previous grades, you have learned the following relationships: $\mathbb{N} \subseteq \mathbb{W} \subseteq \mathbb{Z}$, where $\mathbb{N}$ is the set of natural numbers, $\mathbb{W}$ is the set of whole numbers and $\mathbb{Z}$ is the set of integers.

If $a$ is an integer, then $a = \frac{a}{1}$ and $\frac{a}{1}$ is a rational number.

This implies, every integer is a rational and hence the set of rational numbers includes the set of integers, that is, $\mathbb{Z} \subseteq \mathbb{Q}$.

Thus, we have the following relationships between the set of natural numbers, the set of whole numbers, the set of integers and the set of rational numbers: $\mathbb{N} \subseteq \mathbb{W} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$.These relationships are given in the following Venn diagram.

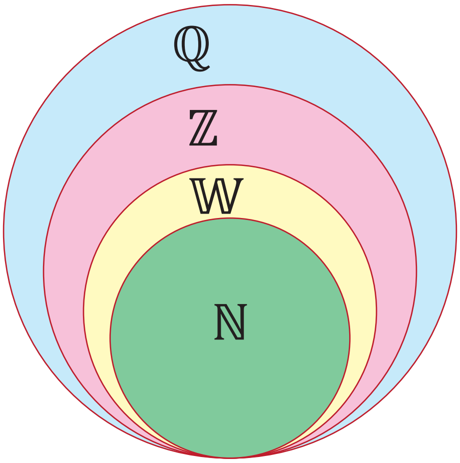



Figure 1.1. Relationships of ℕ, 𝕎, ℤ and ℚ

## Equality of Rational Numbers

### Activity 1.1

Find the values of the variables that make the following fractions equal.

a $\frac{1}{4}=\frac{2}{a}=\frac{b}{16}=\frac{c}{64}=\frac{3}{d}$  b $\frac{e}{7}=\frac{8}{16}=\frac{12}{f}=\frac{g}{56}=\frac{36}{h}$

From your responses in Activity 1.1, observe that two fractions $\frac{a}{b}=\frac{c}{d}$, where $a,b,c$ and $d$ natural numbers, are equal written as $\frac{a}{b}=\frac{c}{d}$, if $a\times d=b\times c$.

The same method can be used to define the equality of two rational numbers.

### Definition 1.2

Two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, where $a, b, c$ and $d$ are integers and $b\neq 0$, $d\neq 0$ are said to be equal, written as $\frac{a}{b}=\frac{c}{d}$, if $a\times d=b\times c$.

### Example 1.2

Show that the rational numbers $\frac{3}{4}$ and $\frac{9}{12}$ are equal.

**Solution**

$3\times 12=36$ and $4\times 9=36$.

This implies $3\times 12=4\times 9$ and hence $\frac{3}{4}=\frac{9}{12}$.

### Example 1.3

Find the value of $a$ so that rational numbers $\frac{a}{3}$ and $\frac{3}{18}$ are equal.

**Solution**

If $\frac{a}{3}=\frac{6}{9}$, then $a\times 9=3\times 6$.

$9a=18$

Therefore, $a=\frac{18}{9}=2$.

**Note:**

Given a rational number $\frac{a}{b}$, if $c$ is a nonzero integer, then $\frac{a}{b} = \frac{a \times c}{b \times c}$.

## Example 1.4

Find another three different forms of the rational number $\frac{3}{4}$.

**Solution**

$\frac{3}{4} = \frac{3 \times 2}{4 \times 2} = \frac{6}{8}$ $\frac{3}{4} = \frac{3 \times 3}{4 \times 3} = \frac{9}{12}$ $\frac{3}{4} = \frac{3 \times 4}{4 \times 4} = \frac{12}{16}$

Therefore, $\frac{6}{8}$, $\frac{9}{12}$ and $\frac{12}{16}$ are three different forms of $\frac{3}{4}$.

## Exercise 1.1

1 For any two integers x and y, is $\frac{x}{y}$ always a rational number? Why?

2 If $\frac{3}{8} = \frac{x}{24}$, then what is the value of $x$?

3 Give at least three different rational numbers which are equal to $\frac{-5}{6}$.

4 Give five rational numbers which are not integers.

5 In a class, 47 students sat for mathematics final examination, 15 of them failed in the examination and the rest passed the examination. Write the ratio of the number students who failed in the examination to the number of students who passed in the examination.

6 Express the number of female students out of total number of students in your class as a rational number.

# 1.2 Decimal and Fraction forms of Rational Numbers

## Activity 1.2

1 Convert each of the following fractions to decimal form.

a $\frac{2}{8}$ b $\frac{13}{6}$ c 2

2 Convert each of the following decimals to fraction form.

a 4.53 b 0.26 c 0.3

From your responses in Activity 1.2, observe that you can convert a number in fraction form into a decimal form and any number in decimal form into fraction.

When a given number in fraction form is changed to decimal form, there are two possibilities:

a the digits after the decimal point terminate; for example, $\frac{5}{4} = 1.25$

b the digits after the decimal point do not terminate, but some of the digits repeat themselves, for example, $\frac{2}{3} = 0.6666\cdots$

### Definition 1.3

A number in decimal form with only finite number of digits after the decimal point is called a terminating decimal.

A number in decimal form with some digits are repeating themselves after the decimal point is called a repeating decimal.

### Example 1.5

a $0.375$, $2.25$ and $0.8$ are examples of terminating decimal numbers.

b $0.666\ldots$, $1.252525\ldots$ and $3.2444\ldots$ are examples of repeating decimals.

## Notation

Repeating decimals can be denoted by putting:

i. a dot at the top of the repeating digit, if there is only one repeating digit, for example, $0.666\ldots = 0.\dot{6}$ and $3.2444\ldots = 3.2\dot{4}$; etc.

ii. a bar at the top of the repeating digits, if there are two or more repeating digits, for example, $1.252525\ldots = 1.\overline{25}$ and $35.132132132\cdots = 35.\overline{132}$; etc.

## Note

To convert fractions to decimals, divide the numerator by the denominator.

### Converting Decimal to Fraction

I. To convert a terminating decimal to fraction form, if there are k digits after the decimal point, then multiply the number by $\frac{10^k}{10^k}$ and you will get a number in fraction form.

## Example 1.6

Convert each of the following decimals to fraction from.

a 0.03 b 1.2 c 13.205

**Solution**

a $0.03 = \frac{0.03 \times 10^2}{10^2}$ (there are two digits after the decimal point)

$= \frac{3}{100}$

b $1.2 = \frac{1.2 \times 10}{10}$ (there is one digit after the decimal point)

$= \frac{12}{10}$

c $13.205 = \frac{13.205 \times 10^3}{10^3}$ (there are three digits after the decimal point)

$= \frac{13205}{1000}$

II. Steps to convert repeating decimals to fractions.

a Represent the repeating decimal number by a variable, say x;

b If there are only k repeating digits but no nonrepeating digits after the decimal point, then multiply both sides by $10^k$.

c Subtract the first equation from the second and then solve for the variable, the result is a fraction form of the number;

d If there are n non-repeating decimals and k repeating decimals after the decimal point, first multiply x by $10^n$ and then by $10^{n+k}$, subtract the second expression from the first expression and solve for x.

## Example 1.7

Convert the number $0.\dot{7}$ to fraction form.

**Solution**

Let $x = 0.777\cdots$ (1)

Since it has only one repeating digit after the decimal point, multiply both sides by 10

$10x = 7.777\cdots$ (2)

Subtract the first equation from the second equation:

$$10x - x = (7.777\cdots) - (0.777\cdots)$$
$$9x = 7$$
$$x = \frac{7}{9}$$

Therefore, $0.77777\ldots = 0.\dot{7} = \frac{7}{9}$

$$\begin{array}{r} 7.777\cdots \\ -\underline{0.777\cdots} \\ 7.0 \end{array}$$

## Example 1.8

Convert the number $4.3\overline{25}$ to fraction form.

**Solution**

Let $x = 4.3\overline{25} = 4.3252525\ldots$ (1)

There are one non-repeating and two repeating digits after the decimal point. Multiply both sides of equation (1) first by 10 and then by $10^3 = 1000$.

$10x = 43.252525\ldots$ (2)

$1000x = 4325.252525\ldots.$ (3)

Subtract the second equation from the third equation

$$1000x - 10x = (4325.252525\ldots.) - (43.252525\ldots)$$
$$990x = 4282$$
$$x = \frac{4282}{990}$$

$$\begin{array}{r} 4325.252525\cdots \\ -\ \underline{43.252525\cdots} \\ 4282 \end{array}$$

Therefore, $4.3252525\ldots = 4.3\overline{25} = \frac{4282}{990} = \frac{2141}{495}$

## Exercise 1.2

1 Convert each of the following fraction to decimal form.

a $\frac{1}{3}$ b $\frac{7}{6}$ c $\frac{119}{20}$

2 Convert each of the following decimals to fraction form.

a $0.\dot{5}$ c $6.\overline{345}$ e 1.35

b $1.\dot{5}$ d $2.3\overline{12}$ f 35.729

3 Show that $2.\dot{9} = 3$

4 Are $0.\dot{3}$ and $\frac{3}{10}$ equal? Why?

## 1.3 Representation of Rational Numbers on the Number Line

### Activity 1.3

Represent each of the following integers on a number line.

a –5 b 0 c 7

To locate an integer $m$ on a number line, first determine the sign of the number (negative or positive).

a if $m$ is a positive integer, then $m$ is located at $m$ units to the right of the point represented by 0, the origin, on the number line.

b if $m$ is a negative integer, then $m$ is located at $-m$ units to the left of point represented by 0, the origin, on the number line.

### Example 1.9

Locate -6 and 3 on a number line.

**Solution**

First draw a number line with an appropriate units and locate point represent by zero.

Then, move 6 units to the left of 0 to locate $-6$ and move 3 units to right of 0 to locate 3 as shown in Figure 1.2

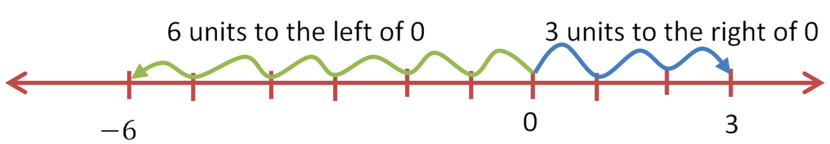



Figure 1.2. A number line

### Locating a Rational number

Consider a rational number $\frac{a}{b}$, where $a$ and $b$ are integers and $b \neq 0$.

Case 1: $\frac{a}{b}$ is a proper fraction, that is, $a$ is closer to 0 than $b$ on the number line.

There are two subcases to consider: $\frac{a}{b} > 0$ and $\frac{a}{b} < 0$.

Subcase 1: Suppose $\frac{a}{b} > 0$.

Divide the line segment with end points represented by 0 and 1 on the number line into $b$ equal parts. The $a^{th}$ point from 0 to 1 is represented by $\frac{a}{b}$.

## Example 1.10

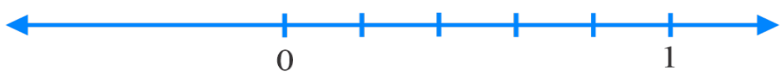
Locate $\frac{4}{5}$ on the number line.

### Solution

$\frac{4}{5}$ is a proper fraction and $\frac{4}{5} > 0$.

**Step 1**: Draw a number line and locate the integers 0 and 1 on the number line.

**Step 2:** Divide the line segment with end points 0 and 1 into 5 equal parts.




**Step 3:** The $4^{th}$ point from 0 to 1 is the point represented by $\frac{4}{5}$ as in the Figure 1.3.

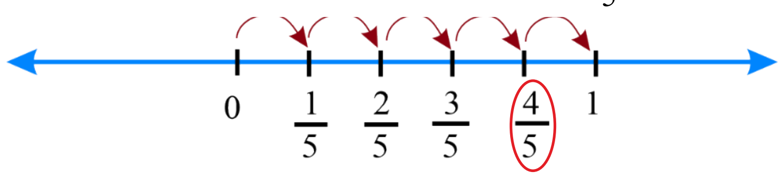



Figure 1.3. Locating rational numbers on the number line

**Subcase 2:** Suppose $\frac{a}{b} < 0$, where $a < 0$ and b > 0.

Divide the line segment with end points represented by $-1$ and $0$ on the number line into $b$ equal parts. The $(-a)^{th}$ point from 0 to $-1$ to the left is represented by $\frac{a}{b}$.

## Example 1.11

Locate $\frac{-3}{4}$ on the number line.

### Solution

$\frac{-3}{4}$ is a proper fraction with $\frac{-3}{4} < 0$

**Step 1**: Draw a number line and locate the integers $-1$ and $0$ on the number line.

**Step 2:** Divide the line segment with end points –1 and 0 into 4 equal parts.

**Step 3:** Staring from 0, move 3 units to the left to –1 and the third point is represented $\frac{-3}{4}$ as shown in Figure 1.4.

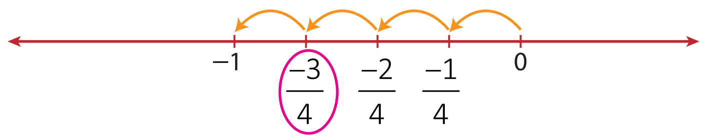



Figure 1.4. Locating rational numbers on the number line

Case 2: $\frac{a}{b}$ is an improper fraction, that is, $b$ is closer to 0 than $a$ on the number line.

Then write $\frac{a}{b} = p + \frac{r}{b}$, where $p$ and $r$ are integers with r is closer to 0 than $b$ on the number line.

In this case also there are two subcases to consider: $\frac{a}{b} > 0$ and $\frac{a}{b} < 0$.

**Subcase 1:** Suppose $\frac{a}{b} > 0$ and $\frac{a}{b} = p + \frac{r}{b}$.

**Step 1**: Draw a number line and locate the points by the numbers p and $p^{+1}$.

**Step 2**: Divide the line segment on the number line with end points represented by $p$ and $p+1$ into $b$ equal parts. Then the $r^{th}$ point from $p$ to $p+1$ is represented by $\frac{a}{b}$.

## Example 1.12

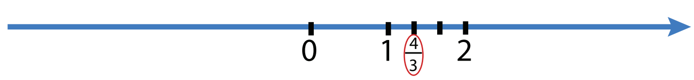
Locate $\frac{4}{3}$ on the number line.

**Solution:**

First write $\frac{4}{3} = 1 + \frac{1}{3}$.

Then divide the line segment with end points 1 and 2 on the number line into 3 equal parts.

Then the first point from 1 to 2 is the point represented by $\frac{4}{3}$. as shown below.




**Subcase 2:** Suppose $\frac{a}{b} < 0$ and $\frac{a}{b} = p + \frac{r}{b}$ with $p < 0, r < 0$ and $b > 0$.

**Step 1**: Draw a number line and locate the points represented by p - 1 and p on the number line.

**Step 2**: Divide the line segment on the number line with end points represented by $p$-1 and $p$ into $b$ equal parts. Then the $(-r)^{th}$ point from $p$ to $p-1$ is represented by $\frac{a}{b}$.

## Example 1.13

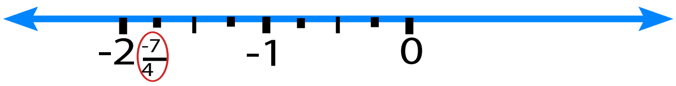
Locate $\frac{-7}{4}$ on the number line.

### Solution

First write $\frac{-7}{4} = -1 - \frac{3}{4}$.

Then divide the line segment with end points −2 and −1 on the number line into 4 equal parts.

Then the third point from −1 and −2 is the point represented by $\frac{-7}{4}$.




### Definition 1.4

Opposite rational numbers are located on the number line in opposite directions from 0 at equal distance. That is, opposite rational numbers are numbers that have the same magnitude, but they are different in signs.

## Example 1.14

Determine the opposites of each of the following rational numbers.

a $-\frac{5}{2}$ b 0.333…

### Solution

a Since $-\frac{5}{2}$ is located to the left of 0 at a distance of $\frac{5}{2}$ from 0 and $\frac{5}{2}$ is located to the right of 0 at equal distance, we conclude that $\frac{5}{2}$ is the opposite of $-\frac{5}{2}$ or vice versa.

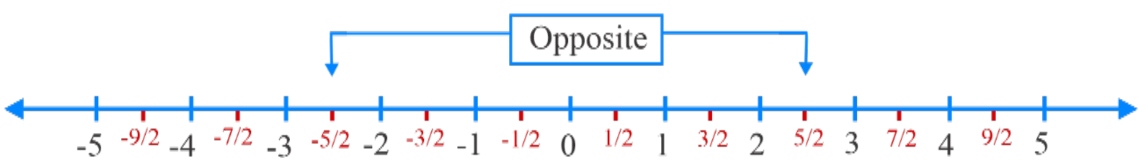



Figure 1.5. Opposite Rational Numbers on the Number Line

b $\frac{1}{3}=0.333\ldots$ is located to the right of 0 and $-\frac{1}{3}=-0.333\ldots$ is located to the left of 0 at equal distance. Thus, the opposite of 0. 333… is $-0.333\ldots$

## Exercise 1.3

1 Insert at least three rational numbers between the following pairs of numbers.

a $-1$ and $1$

b $\frac{1}{3}$ and $\frac{4}{5}$

c $-\frac{1}{3}$ and $-\frac{1}{5}$

d $\frac{3}{2}$ and $2$

e $-\frac{1}{2}$ and $0$

f 0.2 and 0.3

2 Locate the following rational numbers on the number line.

a 3.9

b 2.5

c $-3\frac{14}{5}$

d $\frac{-17}{3}$

e $\frac{5}{6}$

f $\frac{-7}{8}$

3 How many integers are located between $0.\dot{3}$ and $1.5$ on the number line? Why?

4 How many rational numbers are located between $0.\dot{3}$ and $1.5$ on the number line? Why?

# 1.4 Comparing and Ordering Rational Numbers

### 1.4.1 Comparing Rational Numbers

## Activity 1.4

1 Compare the following pairs of fractions.

a $\frac{1}{4}$ and $\frac{2}{4}$

b $\frac{2}{5}$ and $\frac{2}{7}$

2 Compare the following pairs of decimals.

a 0.122 and 0.112

b 1.321 and 2.321

If the two rational numbers are in fraction form you can compare them in two ways:

i. transform the two fractions to decimal forms and compare the decimal forms or

ii. transform the two fractions to fraction forms with the same denominators and compare the numerators.

## Comparison Methods for Rational Numbers

Given two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$

i. if the rational numbers have the same denominator (that is if b = d) and b > 0, then $\frac{a}{b} < \frac{c}{b}$ if $a < c$;

ii. if the rational numbers have different denominators, then convert them to rational numbers having the same denominator and compare the numerators. That is, if b > 0 and d > 0, then $\frac{a}{b} < \frac{c}{d}$ whenever $\frac{a \times d}{b \times d} < \frac{c \times b}{d \times b}$.

Note Any negative rational number is less than any positive rational numbers and any negative rational numbers is less than zero.

### Example 1.15

Compare each of the following pairs of rational numbers.

a $\frac{3}{2}$ and $\frac{5}{2}$

b $\frac{3}{4}$ and $\frac{1}{2}$

c $\frac{7}{9}$ and $-\frac{5}{3}$

d $-\frac{1}{3}$ and 0

e $-\frac{5}{3}$ and $-\frac{1}{2}$

**Solution**

a The rational numbers $\frac{3}{2}$ and $\frac{5}{2}$ have the same denominator. Since $3<5$, we have $\frac{3}{2}<\frac{5}{2}$

b The rational numbers $\frac{3}{4}$ and $\frac{1}{2}$ have different denominators. To compare them, first convert each of them to fractions with the same denominator. Since $\frac{1}{2} = \frac{1 \times 2}{2 \times 2} = \frac{2}{4}$ and $3 > 2$, we have $\frac{3}{4} > \frac{2}{4}$. Therefore, $\frac{3}{4} > \frac{1}{2}$

c Since every negative rational number is less than every positive rational number $-\frac{5}{3} < \frac{7}{9}$

d Since any negative rational number is less than 0, we have $-\frac{1}{3} < 0$

e The rational numbers $-\frac{5}{3}$ and $-\frac{1}{2}$ have different denominators. To compare them, first convert to fractions with the same denominator. $-\frac{5}{3} = -\frac{5 \times 2}{3 \times 2} = -\frac{10}{6}$ and $-\frac{1}{2} = -\frac{1 \times 3}{2 \times 3} = -\frac{3}{6}$

Since $-10 < -3$, we have $-\frac{10}{6} < -\frac{3}{6}$ Therefore, $-\frac{5}{3} < -\frac{1}{2}$

The other way of comparing rational numbers is using place values, after converting the rational numbers to decimal form. To compare two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ where a, b, c and d are integers and $b \neq 0,\ d \neq 0$

**Step 1**: First convert the rational numbers to decimal form by dividing the numerator to the denominator.

**Step 2**: Compare numbers before the decimal point of the two rational numbers.

**Step 3**: If they are equal compare tenth place digit of the first number with tenth place digit of the second number, hundredth place digit with hundredth place digit and so on.

## Example 1.16

Compare $\frac{3}{4}$ and $\frac{7}{8}$ by converting them to decimal form.

### Solution

First convert each rational number to decimals by using long division.

That is $\frac{3}{4} = 0.750$ and $\frac{7}{8} = 0.875$

Since they have equal one's place digit, comparing their tenths' digit, we have $7 < 8$

Therefore, $0.750 < 0.875$

Hence, $\frac{3}{4} < \frac{7}{8}$

## Example 1.17

Compare 31.2168 and 31.2159

### Solution

The digits before the decimal point are equal in numbers and the first two digits after the decimal point are the same.

But the thousands` place digit of the first is 6 and the thousands` place digit of the second number is 5, and also $5 < 6$.

Therefore, $31.2159 < 31.2168$.

## Exercise 1.4

1 Compare each of the following pairs of rational numbers.

a $\frac{1}{4}$ and $\frac{1}{3}$ b $\frac{4}{3}$ and $\frac{4}{5}$ c $-\frac{4}{3}$ and $\frac{16}{20}$

2 Compare each of the following pairs of decimals.

a 10.13 and 11.13 c 13.5746 and 13.5748

b 0.012 and 0.014

3 Compare each of the following pairs of rational numbers by converting them to decimal forms.

a $\frac{2}{3}$ and $\frac{1}{4}$ c $\frac{1}{5}$ and $-\frac{3}{20}$

b $\frac{4}{5}$ and $\frac{1}{5}$ d 0.85 and $\frac{7}{8}$

4 Compare each of the following pairs of rational numbers.

a $\frac{2}{3}$ and 0.56 b 0.6 and $\frac{2}{3}$ c $-1.25$ and $-\frac{4}{3}$

### 1.4.2 Ordering Rational Numbers

## Activity 1.5

1 Locate each of the following fractions on a number line.

$-\frac{1}{2}, \frac{3}{2}, -\frac{2}{3}, \frac{5}{6}$

2 Arrange the numbers in (1) in increasing order.

In order to arrange rational numbers in increasing or decreasing you can use comparison methods. Increasing is also called ascending and decreasing is also called descending. Increasing order means from smaller to larger and descending order means from larger to smaller.

## Example 1.18

a Arrange the rational numbers $\frac{3}{6}, \frac{1}{6}, \frac{5}{6}, -\frac{7}{6}$ in increasing/or ascending order.

b Arrange the rational numbers $\frac{3}{2}, -\frac{5}{4}, \frac{7}{6}, -\frac{13}{12}$ in decreasing/or descending order

### Solution

a Since all the numbers have the same denominator, you can compare the numerators.

That is $-7 < 1 < 3 < 5$

Therefore, $-\frac{7}{6} < \frac{1}{6} < \frac{3}{6} < \frac{5}{6}$ (in increasing order)

b Since the numbers have different denominators, first convert to fractions with the same denominator and compare the numerators of the resulting fractions. That is

$\frac{3}{2} = \frac{3 \times 6}{2 \times 6} = \frac{18}{12}, \quad -\frac{5}{4} = \frac{-5 \times 3}{4 \times 3} = \frac{-15}{12}, \frac{7}{6} = \frac{7 \times 2}{6 \times 2} = \frac{14}{12}$ and

$18 > 14 > -13 > -15$

So, $\frac{18}{12} > \frac{14}{12} > -\frac{13}{12} > -\frac{15}{12}$

Therefore, $\frac{3}{2} > \frac{7}{6} > -\frac{13}{12} > -\frac{5}{4}$ (in decreasing order)

## Exercise 1.5

1 Arrange each of the following rational sets of numbers in increasing/or ascending order.

a $\frac{1}{2}, \frac{1}{3}, \frac{6}{13}, \frac{4}{5}, \frac{7}{18}, \frac{2}{19}$

b $\frac{-1}{3}, \frac{-3}{4}, 0.5, 2, 5, \frac{1}{3}$

2 Arrange the following rational numbers in decreasing/or descending order.

$\frac{3}{8}, 0.\dot{7}, \frac{-4}{5}, \frac{2}{7}, -\frac{22}{7}, 3.14$

## 1.5 Absolute Values of Rational Numbers

From your knowledge in the previous grades, recall that the distance between two points on the number line is the length of the straight path connecting them.

### Activity 1.6

1. a How far is the point represented by 12 from 0 on a number line?

   b How far is the point represented by -12 from 0 on a number line?

2. What do you observe from the questions 1(a ) and 1(b)?

From your responses in Activity 1.6 observe that the distances from the origin to each of the points represented by 12 and -12 are equal and this distance is called the absolute of the numbers.

### Definition 1.5

The absolute value of a rational number x, denoted by $|x|$, is defined by:

$$|x| \begin{cases} x, \text{ if } x > 0 \\ 0, \text{ if } x = 0 \\ -x, \text{ if } x < 0 \end{cases}$$

### Example 1.19

Find the absolute value each of the following numbers.

a $-28$

b $7$

c $-\frac{3}{5}$

d $0$

**Solution:**

a $-28 < 0$ implies $|-28| = -(-28) = 28$

b $7 > 0$ implies $|7| = 7$

c $-\frac{3}{5} < 0$ implies $\left|-\frac{3}{5}\right| = -\left(-\frac{3}{5}\right) = \frac{3}{5}$.

d $|0| = 0$, by definition of absolute value.

## Note:

For any rational number $a$, $|a|$ is the distance from the origion to the point represented by $a$. Thus, $|a| \geq 0$.

## Exercise 1.6

1 Evaluate each of the following.

a $|-13| + |13|$

c $|0|$

e $|9-4| - |-5|$

b $|10-6| + |-4|$

d $|-12| - |12|$

f $-\left|\frac{3}{4}\right|$

2 Find the distance between points represented by the following rational numbers.

a $2$ and $5\frac{2}{3}$

b $-2$ and $5\frac{2}{3}$

**Equations involving absolute values**

## Activity 1.7

1. How many points are located on the number line that are 4 units away from the origin?
2. What are the corresponding number(s) represented these point(s) in (1)?

Geometrically the expression $|x| = 3$ means that the point with coordinate $x$ is 3 units from 0 on the number line. Obviously, the number line contains two points that are 3 units from the origin: one on the right of the origin and the other on the left of the origin. Thus $|x| = 3$ has two solutions x = 3 and x = –3

### Definition 1.6

For any non-negative rational number $a$, $|x| = a$ implies x = $a$ or $x = -a$

## Example 1.20

Solve each of the following equations.

a $|x| = 8$

b $|2x - 1| = 11$

c $|x| = -9$

**Solution**

a $|x| = 8$ implies $x = 8$ or $x = -8$

b $|2x - 1| = 11$ implies $2x - 1 = 11$ or $2x - 1 = -11$

$$2x = 12 \text{ or } 2x = -10$$

$$x = 6 \text{ or } x = -5$$

Therefore, the solutions of the equation $|2x - 1| = 11$ is $x = -5$ or $x = 6$.

c Since the absolute value of any number is non-negative, the equation $|x| = -9$ has no solution. That is, there is no rational number whose absolute value is $-9$.

**Properties of Absolute values**

## Activity 1.8

Compare each of the following pairs of numbers.

a $-3$ and $|-3|$

b $|3|$ and $|-3|$

c $|5|$ and $|-5|$

d $|3 \times (-4)|$ and $|3|\,|-4|$

e $\left|\frac{-3}{4}\right|$ and $\frac{|3|}{|4|}$

f $|3 + 4|$ and $|3| + |4|$

g $|3 + (-4)|$ and $|3| + |4|$

## Note

For any two rational numbers $a$ and $b$ the following relations are true.

a $a \le |a|$

b $|a| = |-a|$

c $|ab| = |a||b|$

d $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, $b \neq 0$

e $|a + b| \le |a| + |b|$

## Example 1.21

i $-3 \le |-3|$ because $|-3| = 3$ and $-3 \le 3$

ii $|3| = |-3|$ because $|-3| = 3$ and $|-3| = 3$

iii $|-2 \times 5| = |-2| \times |5|$ because $|-2 \times 5| = |-10| = 10$ and $|-2| \times |5| = 2 \times 5 = 10$

iv $|-2 \times -5| = |-2| \times |-5|$ because $|-2 \times -5| = |10| = 10$ and $|-2| \times |-5| = 2 \times 5 = 10$

v $\left|\frac{-3}{4}\right| = \frac{|-3|}{|4|}$ because $\frac{3}{4} = \frac{3}{4}$

vi $|-3+5| \leq |-3| + |5|$ because $|-3+5| = 2$ and $|-3| + |5| = 3 + 5 = 8$

## Exercise 1.7

1 Evaluate each of the following expressions.

a $-6x+2|x-3|$ when $x = -3$

b $|m| - m + 3$ when m = 1

c $-2|x - 7|$ when $x = -3$

d $|x - 4| - |5 - y|$ when $x = -2$ and $y = -6$

e $|x|+|y|$ when $x = -3$ and $y = -1$

f $|y| - |x|$ when $y = -7$ and $x = 3$

2 Find the possible value (s) of $x$ that make the given equations true.

a $|2x| = 5$

b $|-x - 1| = 3$

c $|3x - 2| = 4$

d $\left|\frac{4x}{5}\right| = 10$

e $|3x| = 0$

f $|x - 3| = 0$

g $-7|x - 2| = -14$

# 1.6 Operations and their Properties on Rational Numbers

In Grade 7, you have learnt about the four basic operations; addition, subtraction, multiplication and division on integers. In this section you will learn the four basic operations; addition, subtraction, multiplication and division on rational numbers and their properties.

### 1.6.1 Addition and Subtraction of Rational Numbers

## Activity 1.9

1 Find each of the following sums.

a $\frac{1}{2}+\frac{3}{2}$

b $\frac{2}{5}+\frac{5}{3}$

2 Find each of the following differences.

a $\frac{2}{5}-\frac{3}{5}$

b $\frac{4}{7}-\frac{2}{5}$

Adding and subtracting of rational numbers is defined in the same way as addition and subtraction of fractions are defined.

### Definition 1.7

For any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, with $b \neq 0$ and $d \neq 0$:

i the sum of $\frac{a}{b}$ and $\frac{c}{d}$; denoted by $\frac{a}{b}+\frac{c}{d}$ is defined by $\frac{a}{b}+\frac{c}{d}=\frac{(a\times d)+(b\times c)}{b\times d}$

ii the difference of $\frac{a}{b}$ and $\frac{c}{d}$; denoted by $\frac{a}{b}-\frac{c}{d}$ is defined by $\frac{a}{b}-\frac{c}{d}=\frac{(a\times d)-(b\times c)}{b\times d}$

## Example 1.22

Compute each of the following.

a $\frac{3}{8}+\frac{5}{2}$ b $\frac{3}{8}-\frac{5}{2}$ c $-\frac{13}{17}+\left(-\frac{12}{23}\right)$

### Solution

a $\frac{3}{8}+\frac{5}{2}=\frac{(3\times 2)+(8\times 5)}{8\times 2}=\frac{6+40}{16}=\frac{46}{16}$

b $\frac{3}{8}-\frac{5}{2}=\frac{(3\times 2)-(8\times 5)}{8\times 2}=\frac{6-40}{16}=\frac{-34}{16}$

c $-\frac{13}{17}+\left(-\frac{12}{23}\right)=\frac{((-13)\times 23)+(17\times(-12))}{17\times 23}=\frac{(-299)+(-204)}{391}=-\frac{503}{391}$

## Example 1.23

A husband is $32\frac{3}{4}$ years old and his wife is $27\frac{1}{8}$ years old. What is the age difference between the husband and his wife?

### Solution

First change the mixed fractions to improper fractions.

$32\frac{3}{4}=\frac{131}{4}$ and $27\frac{1}{8}=\frac{217}{8}$.

Then $32\frac{3}{2}-27\frac{1}{8}=\frac{131}{4}-\frac{217}{8}$

$=\frac{(131\times 8)-(4\times 217)}{4\times 8}$

$$= \frac{1048 - 868}{4 \times 8}$$

$$= \frac{45}{8} = 5\frac{5}{8}$$

Therefore, the age difference between the husband and his wife is $5\frac{5}{8}$ years.

### Properties of Addition of Rational Numbers

#### Activity 1.10

Compute each of the following and compare your results.

a $\frac{1}{2}+\frac{3}{2}$ and $\frac{3}{2}+\frac{1}{2}$

b $\left(\frac{2}{5}+\frac{5}{3}\right)+\frac{1}{2}$ and $\frac{2}{5}+\left(\frac{5}{3}+\frac{1}{2}\right)$

c $\frac{1}{2}+0$ and $0+\frac{1}{2}$

d $\frac{1}{2}+\left(-\frac{1}{2}\right)$ and $\left(-\frac{1}{2}\right)+\frac{1}{2}$

From your responses in Activity 1.2, observe that the two sums in each of (a), (b), (c) and (d) are equal and these results are true for any group of rational numbers.

Let $\frac{a}{b}, \frac{c}{d}$ and $\frac{e}{f}$, where $b \neq 0, d \neq 0$ and $f \neq 0$, be rational numbers. Then:

i $\frac{a}{b}+\frac{c}{d}=\frac{c}{d}+\frac{a}{b}$ (Addition on the set of rational numbers is commutative)

ii $\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{e}{f}=\frac{a}{b}+\left(\frac{c}{d}+\frac{e}{f}\right)$ (addition on the set of rational numbers is associative)

iii $\frac{a}{b}+0=\frac{a}{b}=0+\frac{a}{b}$ (0 is the additive identity in addition of rational numbers)

iv $\frac{a}{b}+\left(-\frac{a}{b}\right)=0=\left(-\frac{a}{b}\right)+\frac{a}{b}$ ($-\frac{a}{b}$ is the additive inverse of $\frac{a}{b}$

#### Exercise 1.8

1 Compute each of the following.

a $\frac{17}{19}+\frac{22}{9}$

b $\frac{5}{16}-\frac{1}{6}+\frac{7}{6}$

c $-\frac{13}{21}+\frac{6}{15}$

d $-\frac{5}{13}-\left(-\frac{26}{7}\right)$

e $-2+\frac{-5}{24}$

f $0.2+\frac{17}{8}+\frac{8}{9}$

g $-\frac{1}{22}+3\frac{1}{3}$

h $2.\dot{3}-0.\dot{2}$

2 A woman bought 3kg of rice for $57\frac{3}{4}$ Birr, 2kg of wheat for $45\frac{1}{4}$ Birr and 10kg of Teff for $580\frac{1}{2}$ Birr. How much money did she spend in buying all the listed items?

3 Part of a 100m$^2$ farming land is covered by lemon and orange plants. If $\frac{1}{4}$ of the land is covered by orange, $\frac{1}{5}$ of it is covered by lemon and the rest not covered by plants, then

a how much of the farm land is covered by lemon and orange?

b how much of the farm land is not covered by any plant?

4 For any three rational numbers $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$, justify whether each of the following are true or not.

i $\left(\frac{a}{b}-\frac{c}{d}\right)-\frac{e}{f}=\frac{a}{b}-\left(\frac{c}{d}-\frac{e}{f}\right)$

ii $\frac{e}{f}+\left(\frac{a}{b}+\frac{c}{d}\right)=\left(\frac{e}{f}+\frac{a}{b}\right)+\left(\frac{e}{f}+\frac{c}{d}\right)$

iii $\left(\frac{a}{b}+\frac{c}{d}\right)-\frac{e}{f}=\frac{a}{b}+\left(\frac{c}{d}-\frac{e}{f}\right)$

### 1.6.2 Multiplication and Division of Rational Numbers

## Activity 1.11

1 Compute each of the following products.

a $\frac{1}{2}\times\frac{5}{4}$ b $\frac{2}{3}\times\frac{5}{6}$

2 Compute each of the following divisions.

a $\frac{1}{2}\div\frac{5}{4}$ b $\frac{2}{3}\div\frac{5}{6}$

Multiplication and division of rational numbers are defined in the same way as multiplications and divisions of fractions.

### Definition 1.8

For any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, where $b\neq 0,\ d\neq 0$;

i the product of $\frac{a}{b}$ and $\frac{c}{d}$, denoted $\frac{a}{b}\times\frac{c}{d}$, is defined by: $\frac{a}{b}\times\frac{c}{d}=\frac{a\times c}{b\times d}=\frac{ac}{bd}$.

ii the quotient of $\frac{a}{b}$ and $\frac{c}{d}$, (for $c \neq 0$), denoted by $\frac{a}{b} \div \frac{c}{d}$, is defined by:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{a \times d}{b \times c} = \frac{ad}{bc}.$$

## Example 1.24

Compute each of the following operations.

a $-\frac{23}{15} \times \frac{1}{62}$

b $\frac{34}{5} \div \frac{17}{6}$

c $-\frac{23}{15} \times \frac{-33}{62}$

d $-2 \div \frac{28}{37}$

**Solution**

a $-\frac{23}{15} \times \frac{1}{62} = \frac{(-23) \times 31}{15 \times 62} = \frac{-23}{930}$

b $\frac{34}{5} \div \frac{17}{6} = \frac{34}{5} \times \frac{6}{17} = \frac{34 \times 6}{5 \times 17} = \frac{204}{85}$

c $-\frac{23}{15} \times \frac{-3}{62} = \frac{(-23) \times (-3)}{15 \times 62} = \frac{69}{930}$

d $-2 \div \frac{28}{37} = \frac{-2}{1} \times \frac{37}{28} = \frac{(-2) \times 37}{1 \times 28} = \frac{-74}{28}$

### Properties of Multiplication on Rational Numbers:

## Activity 1.12

Compute each of the following and compare your results.

a $\frac{1}{3} \times \frac{2}{5}$ and $\frac{2}{5} \times \frac{1}{3}$

b $\left(\frac{1}{5} \times \frac{2}{3}\right) \times \frac{1}{2}$ and $\frac{1}{5} \times \left(\frac{2}{3} \times \frac{1}{2}\right)$

c $\frac{1}{3} \times \left(\frac{2}{5} + \frac{1}{2}\right)$ and $\left(\frac{1}{3} \times \frac{2}{5}\right) + \left(\frac{1}{3} \times \frac{1}{2}\right)$

d $\left(\frac{2}{5} + \frac{1}{2}\right) \times \frac{1}{3}$ and $\left(\frac{2}{5} \times \frac{1}{3}\right) + \left(\frac{1}{2} \times \frac{1}{3}\right)$

From your responses in Activity 1.14, observe that the two results in each case are equal.

Let $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$, where $b \neq 0$, $d \neq 0$ and $f \neq 0$, be rational numbers. Then

i $\frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b}$ (multiplication of rational numbers is commutative) ii $\left(\frac{a}{b}\times\frac{c}{d}\right)\times\frac{e}{f}=\frac{a}{b}\times\left(\frac{c}{d}\times\frac{e}{f}\right)$ (multiplication of rational numbers is associative)

iii $\frac{a}{b}\times\left(\frac{c}{d}+\frac{e}{f}\right)=\left(\frac{a}{b}\times\frac{c}{d}\right)+\left(\frac{a}{b}\times\frac{e}{f}\right)$ (multiplication is distributive over addition in the set of rational numbers)

iv $\left(\frac{a}{b}+\frac{c}{d}\right)\times\frac{e}{f}=\left(\frac{a}{b}\times\frac{e}{f}\right)+\left(\frac{c}{d}\times\frac{e}{f}\right)$ (multiplication is distributive over addition in the set of rational numbers)

v $\frac{a}{b}\times 1=\frac{a}{b}=1\times\frac{a}{b}$ (1 is the multiplicative identity in the set of real numbers)

## Note

If $\frac{a}{b}$ is a rational number with $a \neq 0$ and $b \neq 0$, $\frac{a}{b}\times\frac{b}{a}=\frac{ab}{ba}=\frac{ab}{ab}=1$ . Thus $\frac{b}{a}$ is called the multiplicative inverse of $\frac{a}{b}$ and denoted by $\left(\frac{a}{b}\right)^{-1}$

That is, $\left(\frac{a}{b}\right)^{-1}=\frac{b}{a}$.

### Example 1.25

Find the multiplicative inverse of each of the following numbers.

a $\frac{3}{5}$ b $\frac{-7}{6}$ c 0.25

**Solution**

a $\left(\frac{3}{5}\right)^{-1}=\frac{5}{3}$

b $\left(\frac{-7}{6}\right)^{-1}=\frac{-6}{7}$

c $0.25=\frac{1}{4}$ and $(0.25)^{-1}=\frac{4}{1}=4$

### Exercise 1.9

1 Compute each of the following.

a $\frac{4}{5}\times\frac{12}{7}$

b $\frac{13}{5}\div\frac{-17}{4}$

c $-\frac{7}{8}\times\frac{14}{9}\times\frac{4}{3}$

d $\frac{-15}{6}\div(-24)$

e $4\frac{2}{3}\times\left(-2\frac{1}{3}\right)$

f $-4\frac{1}{3}\div\frac{2}{3}$

g $-\frac{7}{8}\times\frac{14}{9}\times\frac{4}{3}$

h $1.5\div\frac{5}{2}$

2 If the product of two numbers is $-\frac{28}{27}$ and one of the number is $-\frac{4}{9}$, then find the other number.

3 Compute each of the following

a $\frac{-7}{16}+\frac{13}{5}\times\frac{-17}{20}$ b $\frac{-2}{3}\times\left(\frac{-1}{2}-\frac{3}{2}\right)$ c $1.5\div\frac{5}{2}\times 0.2-0.3+0.7$

4 Find the multiplicative inverse of each of the following numbers.

a $\frac{9}{11}$ b $\frac{-25}{29}$ c 1.35 d −21.3

5 Five students divided 1.45 meters long sugarcane into five equal parts. How many meters of sugarcane does each student get?

6 A photograph measuring 8cm by 4cm is enlarged by a ratio of 11:4. What are the dimensions of the new photo?

## 1.7 Applications of Rational Numbers in Calculating Interest and Loans

There are different real life situations involving the set of rational numbers. Some of these situations are related to shares, interests, loans, etc. In this section, you will learn some applications of rational numbers, simple interest and compund interest.

### 1.7.1 Simple Interest

Amount of money that a person or a company borrows from a bank or a financial institution to his/her needs is called a loan. Some examples of loan are home loans, car loans, education loans, personal loans etc. A loan is required to be returned by the person or the company that borrows it to the financial institution on time with an extra amount, which is called the interest of the loan.

The interest could be simple interest; it is an amount that will be paid based on a certain rate only on the principal amount borrowed from the given institution. In simple interest, the principal amount is always the same.

Interest could also be paid for individuals for saving money in banks or financal institutions.

**Activity 1.13**

Suppose your father saves 1000 Birr every month with a rate of 7% per year in a certain microfinance. If the interest is calculated only for the principal amount;

a what will be the total interest in one year?

b what will be the total amount at the end of two years?

From your responses in Activity 1.13, observe that, to determine the amount of interest for a given principal amount, you have to multiply the given principal amount with the interest rate and with the given amount of time.

### Definition 1.9

If a principal amount P is invested with a simple interest rate R in T years, then the total interest I after T years is given by:

$$I = PRT$$

The total amount A after T years is the principal plus the interest and it is given by:

$$A = I + P$$

## Example 1.26

A textile factory takes a loan of 100,000 Birr from microfinance for a period of 2 years with a simple interest rate of 10% per year. Find the total interest and the total amount the factory has to pay at the end of the two years.

### Solution

The principal amount is P = 100,000 Birr, the Interest Rate is R = 10% and the number of years is T = 2 Years.

The amount of interest that has to be paid in 2 years is given by

$$I = \text{PRT} = 100,000\text{Birr} \times 10\%\text{per Year} \times 2 \text{ Years}$$

$$= 100,000 \times 0.1 \times 2$$

$$= 20,000 \text{ Birr}$$

The total amount that the factory las to play to the microfinance at the end of two years will be

$$A = P + I$$

$$= 100,000\,\text{Birr} + 20,000\,\text{Birr}$$

$$= 120,000\,\text{Birr}.$$

## Example 1.27

A farmer paid a total amount of 9100 Birr to the amount 7000 Birr which he borrowed from a certain microfinance with a simple interest for 2 years. Find the rate of interest.

### Solution

Given $A = 9100\,\text{Birr}$, $P = 7000\,\text{Birr}$ and $T = 2$ years.

You are required to find I and R

$$I = A - P = 9100 \text{ Birr} - 7000 \text{ Birr}$$

$$= 2100 \text{ Birr}$$

And, $I = PRT$ implies

$$R = \frac{I}{PT} = \frac{2100\,\text{Birr}}{7000\,\text{Birr} \times 2} \times 100\%$$

$$= \frac{2100}{14000} \times 100\% = 15\%$$

Therefore, the rate of the simple interest was 15%.

## Exercise 1.10

1. Find the principal which earns Birr 115.38 in 8 years at a rate 4% simple interest per year.
2. Find the time in which Birr 1680 will earn simple interest of Birr 290 at a rate of 5% per year.
3. Find the rate per year at which Birr 380 earns simple interest of birr 128.25 in 7 years and 6 months.
4. A private limited company borrows Birr 800,000 for 2 years at a simple interest rate of 12%. What is the total amount that must be repaid at the end of two years?

### 1.7.2 Compound Interest

## Activity 1.14

Suppose your mother saved 3000 Birr in a bank with interest rate 7% per year.

a. Calculate the simple interest after 2 years.

b. If the interest is calculated for both saved amount and interest added on each year, calculate the interest after 2 years.

From your responses in Activity 1.14, Question (b), observe that, interest is paid for the principal amount and interest earned during the previous period; the interest paid in such cases is called compound interest.

### Definition 1.10

If P amount is invested at a rate of r% compounded annually, the compound interest at the end of the $n^{th}$ year is computed by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where $A$ is the total amount, $p$ is principal, $r$ is interest rate, $t$ is time and $n$ is number of times interest is compounded per unit time.

## Example 1.28

Find the amount when Birr 2000 is invested for 3 years with an interest of 6% compounded annually.

### Solution

Since it is a compound interest, we must find the interest at the end of each year and add it to the principal before computing the interest for the next years.

$1^{st}$ year interest $I = PRT = 2000 \times 0.06 = 120$ Birr

The Amount at the end of the $1^{st}$ year is 2000 Birr + 120 Birr= 2120 Birr

$2^{nd}$ year Interest $I = PRT = 2120\,\text{Birr} \times 0.06 = 127.20\,\text{Birr}$

The Amount at the end of the 2nd year is 2247.20 Birr

$3^{rd}$ year interest $I = PRT = 2447.20\,\text{Birr} \times 0.06 = 134.80\,\text{Birr}$

Therefore, the Amount at the end of the $3^{rd}$ year is 2382 Birr

Using the formula for compound interest, this can be calculated as:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$= 2000 Birr\left(1 + \frac{6}{100 \times 1}\right)^{1\times 3}$$

$$= 2000 Birr\left(1 + 0.06\right)^{3}$$

$$= 2000 Birr\left(1.06\right)^{3} \approx 2382\text{Birr}$$

## Example 1.29

Find the amount at the end of $5^{th}$ year if 2000 Birr is borrowed at 5% interest compounded annually.

### Solution

Given $P = 2000$ Birr, $r = 5\%$, $t = 5$ years, $n = 1$

Required $A = ?$

$$A = P\left(1 + \frac{r}{n}\right)^{nt} = 2000\,\text{Birr}\left(1 + \frac{5}{100}\right)^{5}$$

$$= 2000\,\text{Birr}\left(1 + 0.05\right)^{5}$$

$= 2000 \text{ Birr} (1.05)^5 = 2552.56 \text{ Birr}$

Therefore, the amount at the end of 5th year is 2552.56 Birr.

## Exercise 1.11

1. Find the amount of 50,000 Birr at a rate of 12% compounded annually at the end of 3 years.
2. A company increases its capital by 10% each year. If it starts with 100,000 Birr capital, how much Birr will the company have at the beginning of the fourth year?
3. A small scale enterprise borrows 100,000 Birr to start business. The enterprise borrows the money at 15% interest and repays it in full after three years. How much interest will the enterprise pay?
4. W/ro Emebet saved 1000 Birr in Commercial bank of Ethiopia at 7% interest rate compounded annually. How much money will she get at the end of 5 years?

# Unit Summary

1 Terminating and repeating decimals are rational numbers.

2 Negative rational numbers are located to left of zero on the number line, whereas positive rational numbers are located to right of zero on the number line.

3 $\mathbb{Z}$ is closed under the operations "+", "−" & "×" , but not "÷".

4 $\mathbb{Q}$ is closed under the operations "+", "−" & "×" , but not "÷".

5 The set of nonzero rational numbers is closed under the operations "+", "−" , "×" & "÷".

6 $|x| = |-x|$ for all rational numbers. $x \in \mathbb{Q}$

7 For any rational numbers $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$,

a $\frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd}$

b $\frac{a}{b} - \frac{c}{d} = \frac{ad - cb}{bd}$

c $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$

d $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$

8 The sum of two opposite rational numbers is 0

9 For any rational numbers $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in \mathbb{Q}$ when $b, d, f \neq 0$, we have

a $\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$

b $\frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b}$

c $\frac{a}{b} \times \left(\frac{c}{d} + \frac{e}{f}\right) = \left(\frac{a}{b} \times \frac{c}{d}\right) + \left(\frac{a}{b} \times \frac{e}{f}\right)$

d $\left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f} = \frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right)$

e $\left(\frac{a}{b} \times \frac{c}{d}\right) \times \frac{e}{f} = \frac{a}{b} \times \left(\frac{c}{d} \times \frac{e}{f}\right)$

10 Rules of signs for Addition:

Let $a$ and $b$ are rational numbers, then

a $a+b>0$ if either both are positive or one of the summands is greater than opposite of the other

b $a+b<0$. if either both are negative or one of the summands is less than opposite of the other.

11 Rules of signs of multiplication:
Let $a$ and $b$ are rational numbers, then

a If $a>0$ and $b>0$ ,then $ab>0$

b If $a\langle 0$ and $b\rangle 0$, then $ab<0$

c If $a>0$ and $b<0$, then $ab<0$

d If $a<0$ and $b<0$, then $ab>0$

12 Rules of signs of division:

Let $a$ and $b$ $(b\neq 0)$ are rational numbers, then

a If $a>0$ and $b>0$, then $\frac{a}{b}>0$

b If $a>0$ and $b<0$, then $\frac{a}{b}<0$

c If $a<0$ and b $>0$, then $\frac{a}{b}<0$

d If $a<0$ and $b<0$ then $\frac{a}{b}>0$

13 For any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$

$\frac{a}{b}=\frac{c}{d}$ if $ad=bc$

14 The simple interest $I=PRT$ is computed for only saved or deposited amount, whereas the compound interest $A=P\left(1+\frac{R}{n}\right)^{nT}$ is computed for both saved or deposited amount and interest added, where $P$ is principal, $R$ is interest rate, $T$ is time and $n$ is number of times the interest is compounded per unit time.

## Review Exercises

1 Convert each of the following decimals to fraction form.

a $56.42358$

b $6.\overline{73}$

c $.32\overline{56}$

d $\frac{2}{3} + 0.\dot{2}$

2 Locate each of the following rational numbers on the number line.

a $\frac{7}{3}$

b $-\frac{2}{5}$

c $1.2$

d $1.\dot{6}$

3 Arrange the following sets of rational numbers in ascending order.

a $-\frac{2}{3}, \frac{27}{9}, -\frac{5}{3}, 0, \frac{1}{2}$

b $1.25, \frac{3}{2}, 1.25\frac{23}{30}, -\frac{23}{5}, -\frac{17}{6}$

c $6.0, 0.6, 0.66, 0.606, -0.06, -6.6, 6.606$

4 Insert at least three rational numbers between the of the following pairs of rational numbers.

a $\frac{5}{13}$ and $\frac{7}{8}$

b $\frac{6}{11}$ and $\frac{5}{7}$

c $\frac{-4}{7}$ and $\frac{-3}{5}$

5 Evaluate each of the following operations.

a $-4(5-(36\div4))$

b $10-(5-(4-(8-2)))$

c $3\frac{1}{5} + \left(\frac{-7}{8}\right)$

d $\frac{-1}{4} + \frac{-5}{9}$

e $1.\dot{4} + 0.\dot{5}$

6 Find the simplified form of $\left[\left(\frac{1}{2}+\frac{1}{3}\right)\times\frac{1}{4}\right]\div\left[\left(\frac{2}{5}+\frac{3}{4}\right)\div\frac{6}{12}\right]$

7 In a Physics text book, 35% of the pages are colored. If there are 98 colored pages, how many pages are there in the whole text book?

8 If $\frac{2a-3}{5}$ and $\frac{7}{8}$ are equal rational numbers, then what is the value of $a$?

9 Suppose you want to borrow 5,000 Birr at 15% interest per year from Abay Bank for 6 months. How much interest would you pay to the bank? How much is the total money you pay to the bank at the end of 6 months?

10 W/ro. Abeba borrowed 6000Birr from a certain Micro-finance at the rate of 15% compounded annually. How much will she pay at the end of the fourth year?

# Unit 2

# SQUARES, SQUARE ROOTS, CUBES AND CUBE ROOTS

## Learning outcomes:

**After completing this unit, you will be able to:**

- understand the notion of square, square root, cube and cube root.
- determine the square and cube of rational numbers.
- determine the square roots and cube root of rational numbers.
- approximate square/cube roots of rational numbers by using table values and scientific calculators.
- apply the concept of squares, square roots, cubes and cube roots in the real-life problems.

## *Key terms*

- square
- perfect square
- square roots
- approximate value
- table value
- radical sign
- decimal place
- cube
- perfect cube
- cube root
- closest whole number
- scientific calculator
- base
- exponent

# Introduction

In your previous grades you have been working with numbers. When working on numbers you have seen how to multiply a number by itself, which is squaring the number; finding a number whose square is a certain given number, finding a square root. You have also learned also how to multiply a number by itself three times, which is called cubing the number and finding a number whose cube is a certain given number.

In this unit you will learn squaring a number, finding square root of a number, cubing a number and finding cube root of a given number.

Squares, square roots, cubes and cube roots of numbers are commonly used in banking, physics and geometry.

## 2.1 Squares and Square Roots

### 2.1.1 Square of a Rational Number

**Activity 2.1**

A farmer has planted avocado trees in a square pattern as shown in Figure 2.1 below.

Figure 2.1.

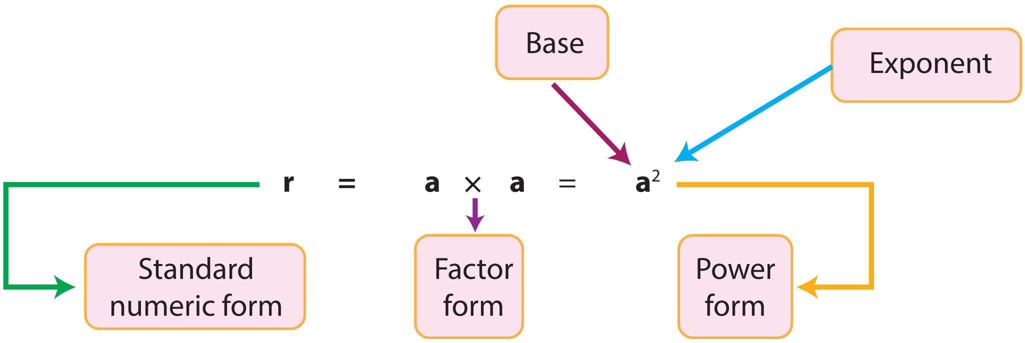
If there are 20 rows and 20 avocado trees in each row, find the total number of avocado trees that the farmer has planted.

In Activity 2.1, to determine the total number of avocado trees that the farmer has planted, you have to multiply the number of rows, which is 20, with the number of avocado trees in each row, which is again 20. In the process, you have to multiply 20 by 20, multiplying 20 by itself. This process is called squaring a number.

## Definition 2.1

The process of multiplying a number by itself is called squaring the number. The product of a number y and itself is called the square of the number, denoted by $y \times y = y^2$ and read as "y squared" or "y to the power of 2".

In the expression below, for two rational numbers a and r if is the square of a is r, then is called the base (the number to be multiplied with itself) and 2 is called an exponent (number of times that the number appears in the product).




## Example 2.1

Show that each of the following numbers is a square number: 0, 1, 4, 9, 16, 25.

**Solution**

$0^2 = 0 \times 0 = 0$ $\quad 1^2 = 1 \times 1 = 1,$ $\quad 2^2 = 2 \times 2 = 4$

$3^2 = 3 \times 3 = 9$ $\quad 4^2 = 4 \times 4 = 16$ $\quad 5^2 = 5 \times 5 = 25$

This implies, 0, 1, 4, 9, 16, 25 are square numbers.

## Activity 2.2

Complete the following table.

| y | 1 | 3 | 4 | 5 | 6 | 7 | 9 | 12 | 13 | 16 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| y+y | 2 | 6 | 8 | 10 | 12 | | | | | | |
| y×y | 1 | 9 | 16 | 25 | 36 | | | | | | |

Explain the differences between $y + y$ and $y \times y$ .

From your responses in Activity 2.2, observe that for each of the given numbers $y + y$ and $y \times y$ are different. When a number is multiplied by itself we get another number which is the product of two equal numbers, for example $3 \times 3 = 3^2 = 9$. On the other hand when a number is added onto itself, we get another number which is the sum of two equal numbers, for example, $3 + 3 = 2 \times 3 = 6$.

## Note

In general, for a number y, $y + y = 2y$ and $y \times y = y^2$ are different

### Definition 2.2

A whole number $y$ is called a perfect square or a square number if it is the square of a certain whole number $x$, that is $y$ and x are whole numbers and $y = x^2$

## Example 2.2

Show that the whole numbers 0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169 are perfect squares and the whole numbers 12, 23 and 232 are not perfect squares.

**Solution**

$0^2 = 0 \times 0 = 0$, $1^2 = 1 \times 1 = 1$, $2^2 = 2 \times 2 = 4$ $3^2 = 3 \times 3 = 9$,

$4^2 = 4 \times 4 = 16$, $5^2 = 5 \times 5 = 25$, $6^2 = 6 \times 6 = 36$, $7^2 = 7 \times 7 = 49$,

$8^2 = 8 \times 8 = 64$ $9^2 = 9 \times 9 = 81$, $10^2 = 10 \times 10 = 100$, $11^2 = 11 \times 11 = 121$,
$12^2 = 12 \times 12 = 144$ $13^2 = 13 \times 13 = 169$.

This implies the numbers 0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169 are perfect squares.

i. $3^2 = 9$, $4^2 = 16$ and $9 < 10 < 16$ implies there is no whole number whose square is 12.

ii. $4^2 = 16$, $5^2 = 25$ and $16 < 23 < 25$ implies there is no whole number whose square is 23.

iii. $15^2 = 225$, $16^2 = 256$ and $225 < 232 < 256$ implies there is no whole number whose square is 232.

Therefore, the whole numbers 12, 23 and 232 are not perfect squares.

### Use of Prime Factorization to Determine Perfect Squares

## Activity 2.3

1 Find the prime factorization of each of the following numbers.

a 36 c 400

b 194 d 1000

2 Which of the numbers in Question 1 above can be written as a product of two identical sets of prime factors?

In your responses in Activity 2.3, observe that 36 can be written as a product of two identical sets of prime factors and 400 also can be written as a product of two identical sets of prime factors. The numbers 36 and 400 are both perfect squares.

You can use prime factorization of a given whole number to determine whether the given whole number is a perfect square or not. The following steps can be used for this purpose.

**Step 1:** First find the prime factorization of the number;

**Step 2:** if possible, arrange the factors so that the number is a product of two identical sets of prime factors;

**Step 3:** if Step 2 is possible, then the given number is a perfect square and if Step 2 is not possible, then the number is not a perfect square.

## Example 2.3

Use prime factorization to determine if the following numbers are perfect squares.

a 144

b 1250

c 62500

### Solution

a $144 = 2 \times 2 \times 2 \times 2 \times 3 \times 3 = (2 \times 2 \times 3) \times (2 \times 2 \times 3) = 12 \times 12 = 144$. Since the factors can be arranged so that 144 is a product of two identical set of prime factors, $(2 \times 2 \times 3) \times (2 \times 2 \times 3)$, we conclude that 144 is a perfect square.

b $1250 = 5 \times 5 \times 5 \times 5 \times 2$. Since the factors cannot be arranged as a product of two identical set of prime factors, 1250 is not a perfect square. 1250 is not a perfect square

c $62500 = (5 \times 5 \times 5 \times 2) \times (5 \times 5 \times 5 \times 2) = 250 \times 250 = (250)^2$. Since the factors can be arranged as a product of two identical set of prime factor, $(5 \times 5 \times 5 \times 2) \times (5 \times 5 \times 5 \times 2)$ the number 62500 is a perfect square.

## Note

1 If a given whole number is a square number, then its unit's digit is either 0,1,4,5,6 or 9.

2 No square whole number ends with an odd number of zeros, for example 10, 1000, and 100000 are not square numbers.

### The Square of a Rational Number in Fraction Form

In Unit 1, you have learned that, rational numbers can be written in the form $\frac{a}{b}$, where $a$ and $b$ are integers and $b \neq 0$. Therefore, squaring the rational number $\frac{a}{b}$ is:

$$\left(\frac{a}{b}\right)^2 = \frac{a}{b} \times \frac{a}{b} = \frac{a \times a}{b \times b} = \frac{a^2}{b^2}.$$

## Example 2.4

Find the square of each of the following rational numbers.

a $\frac{1}{3}$ b $-\frac{6}{5}$ c $\frac{10}{11}$ d $\frac{20}{19}$

**Solution**

a $\left(\frac{1}{3}\right)^2 = \frac{1^2}{3^2} = \frac{1}{9}$

b $\left(-\frac{6}{5}\right)^2 = \frac{(-6)^2}{5^2} = \frac{36}{25}$

c $\left(\frac{10}{11}\right)^2 = \frac{10^2}{11^2} = \frac{100}{121}$

d $\left(\frac{20}{19}\right)^2 = \frac{20^2}{19^2} = \frac{400}{361}$

### The Square of a Rational Number in Decimal Form

If a rational number is written in decimal form you can find the square of the number by multiplying the number by itself.

## Example 2.5

Find

a $(3.24)^2$ b $(23.7)^2$

**Solution**

a
```
   3.24
 × 3.24
  1296
 648
972
10.4976
```

b
```
   23.7
 × 23.7
  1659
 711
474
561.69
```

Therefore, $(3.24)^2 = 10.4976$ and $(23.7)^2 = 561.69$.

## Exercise 2.1

1 Find the square of each of the following numbers.

a 17
b -8
c 0.9
d $\frac{1}{6}$

2 Express the following numbers in power form and identify the base and exponent.

a 36
b 64
c 0.49
d $\frac{1}{100}$

3 Which of the following numbers are perfect squares?

a 0.09
b 0.9
c 0.81
d 8
e 0.25
f -4
g 2
h 81
i 0.3
j 0.04

4 Find two whole numbers whose squares have a sum of 45.

5 List all square numbers less than 100.

6 Show that the difference between the $7^{th}$ square whole number and the $4^{th}$ square whole number is a multiple of 3.

7 Show that the difference between any two consecutive square natural numbers is an odd natural number.

### Use of Table of Squares

In the previous discussion, you have seen that the square of a rational number is a nonnegative rational number and you can determine the square of a rational number by applying the usual procedure of multiplication.

Applying the usual procedure of multiplication to find the square of a rational number is sometimes tedious and time consuming. For this reason, tables of squares of numbers are prepared. In such Tables of squares find the page with the formula $y = x^2$. The first column is headed by "$x$", which lists the numbers from 1.0 to 9.9 and the first row contains numbers from 0 to 9.

Now if you want to determine the square of a number from the Tables of squares, the procedures are illustrated with the following example.

## Example 2.6

Find $(3.85)^2$ from the Table of squares.

### Solution

**Step 1:** In the first column which starts with $x$ find the row which starts with 3.8;

**Step 2:** In the first row, which starts with $x$, find the column which starts with 5;

**Step 3:** Read the number in the intersection of the row which starts with 3.8 and the column which starts with 5, that is 14.82, hence $(3.85)^2 \approx 14.82$.

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 | | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.0 | | | | | | | | | | |
| .<br>.<br>. | | | | | | | | | | |
| 3.8 | | | | | | 14.82 | | | | |
| .<br>.<br>. | | | | | | | | | | |
| 9.9 | | | | | | | | | | |

Figure 2.2. Table of Squares

## Note

1 The steps 1 to 3 can be shortened by "3.8 under 5;

2 The values you get from table of squares are mostly approximates.

3 In the Table of squares, only the squares of numbers from 1.00 to 9.99 are given.

As you have learned in Unit 1, a rational number can be written as a number between 1 and 10 and powers of 10, for example $2345 = 2.345 \times 1000$. To find the square of a number that is not in this range by using Table of squares first write the number as product of a number between 1 and 10 and a power of 10.

## Example 2.7

Complete each of the following using Table of squares.

a $(32.4)^2$ b $(567)^2$

### Solution

a $32.4 = 3.24 \times 10$ and from the Table of squares $(3.24)^2 \approx 10.50$.

Therefore, $(32.4)^2 = (3.24)^2 \times 10^2 \approx 10.50 \times 100 = 1050$.

b $(567)^2 = (5.67)^2 \times 100^2$ and from Table of squares, $(5.67)^2 \approx 32.149$.

Therefore, $(567)^2 = (5.67)^2 \times 100^2 \approx 32.149 \times 10000 = 321490$.

### Use of Scientific Calculators

You can use scientific calculators to find squares of numbers. Consider a scientific calculator as shown in Figure 2.3.

Steps on how to use a scientific calculator to determine square of a given number:

**Step 1:** Write the number on a scientific calculator, by pressing the digits on the calculator;

**Step 2:** Press the square sign $x^2$ on the calculator;

**Step 3:** Round the result to the desired decimal places to approximate the given square number whenever necessary.

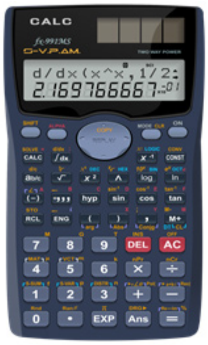


Figure 2.3. Scientific Calculator

## Example 2.8

Approximate each the following numbers to two decimal places using scientific calculators.

a $(3.15)^2$ b $\left(\frac{2}{3}\right)^2$ c $(3.132)^2$

### Solution

Using your scientific calculator, first press the digits of the given number and then the square sign ($x^2$) gives the required result and finally approximate the given number to the nearest two decimal places.

a $(3.15)^2 = 9.9225 \approx 9.92$

b First, change $\frac{2}{3}$ to decimal form: $\frac{2}{3} = 0.666\ldots$

Then $(0.666\ldots)^2 = 0.444\ldots \approx 0.44$

c $(3.132)^2 = 9.809424 \approx 9.81$

## Exercise 2.2

1 Find each of the following numbers using Table of squares.

a $(8.54)^2$

b $(35.42)^2$

c $(0.151)^2$

2 Approximate each of the following numbers using a scientific calculator.

a $(3.58)^2$

b $(14.68)^2$

c $(9230)^2$

### 2.1.2 Square Root of a Rational Number

## Activity 2.4

1 A woman wants her square bedroom to have a room area of 36 square meters. Find the length of each side of the bedroom.

2 Find a number whose square is:

a 1

b 9

c 36

d 0.01

e $\frac{4}{9}$

f 64

In your responses in Activity 2.4, you need to find a number whose square is the given number. Finding the number whose square is given is known as finding the square root of the number.

### Definition 2.3

Let y $\geq 0$ and x $\geq 0$ be rational numbers. If y is the square of the number x, that is, $y=x^2$, then x is called the square root of y. This can be written symbolically as $x=\sqrt{y}$.

In the notation, the symbol $\sqrt{\ }$ is called the radical sign, y is called radical.

## Example 2.9

a $\sqrt{0}=0$, because $0^2=0$,

b $\sqrt{9}=3$, because $3^2=9$

c $\sqrt{0.36}=0.6$, because $(0.6)^2=0.36$.

d $\sqrt{49}=7$, because $7^2=49$

## Square Roots of Perfect Squares

You can use prime factorization to find the square roots of perfect square of natural numbers.

**Step 1:** Write the given number as a product of two identical products of prime factors;

**Step 2:** one of the two identical products of prime factors is the square root of the given number.

### Example 2.10

Use prime factorization to determine each of the following numbers.

a $\sqrt{169}$ b $\sqrt{400}$ c $\sqrt{625}$

**Solution**

a The prime factorization of 169 is $169=13\times13$ and the two identical prime factors are 13 and 13. Therefore, $\sqrt{169}=13$.

b The prime factorization of 400 is $400=2^4\times5^2=\left(2^2\times5\right)\times\left(2^2\times5\right)$ and the two identical products of prime factors are $2^2\times5$ and $2^2\times5$. Therefore, $\sqrt{400}=2^2\times5=20$.

c The prime factorization of 625 is $625=5^2\times5^2$ and the two identical products of prime factors are $5\times5$ and $5\times5$. Therefore, $\sqrt{625}=5\times5=25$.

### Exercise 2.3

1 Complete the following table.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $a^2$ | 4 | 9 | 16 | 25 | 36 | 49 | 0.04 | 0.09 | 0.16 |
| $a$ | 2 | | | | | | | | |

2 Which the following numbers are perfect squares?

a 100

b 144

c $\frac{1}{16}$

d $\frac{9}{25}$

e 8

f 3

g $\frac{6}{16}$

h 0.9

3 List the square of all prime numbers less than 30.

4 Compute the following square roots.

a $\sqrt{1}$

b $\sqrt{49}$

c $\sqrt{9}$

d $\sqrt{\frac{9}{16}}$

e $\sqrt{0.64}$

5 Find the square roots of each of the following numbers.

a 100

b 900

c 121

d 10000

e 729

f 841

6 A checkerboard is a large square that is made up of 32 small white squares and 32 small black squares. How many small squares are along one side of the checkerboard?

7 121 grade eight students need to be seated in square form for sport festival ceremony. How many students should be in each row?

## 2.2 Approximation of Square Roots of Rational Number

In the previous discussion, we have seen how to determine square roots of perfect squares. But most of practical problems involve numbers that are not perfect squares. In such cases, we approximate square roots of numbers. There are different ways to approximate square roots. The most common ones are using Table of squares and scientific calculator.

### Use of Scientific Calculators to Approximate Square Roots

You can use scientific calculators to find square roots of numbers. Consider a scientific calculator as shown in Figure 2.4.

Steps on how to use a scientific calculator to determine square root of a given number:

**Step 1:** Write the number on a scientific calculator;

**Step 2:** Press the square root sign $\sqrt{}$ on the calculator;

**Step 3:** Round the result to the desired decimal places to approximate the given square root.

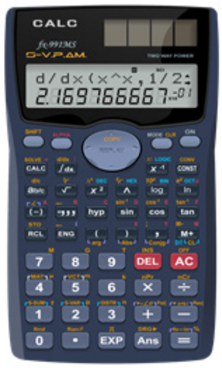



Figure 2.4. Scientific Calculator

## Example 2.11

Approximate each of the following to two decimal points using scientific calculators.

a $\sqrt{2.6}$ b $\sqrt{28}$ c $\sqrt{62}$

### Solution

Press the number first and then the square root sign on the scientific calculator, which gives you the result. Next, by rounding to two decimal places, we obtain:

a $\sqrt{2.6}=1.612451\ldots\approx 1.61$

b $\sqrt{28}=5.291502\ldots\approx 5.29$

c $\sqrt{62}=7.874007\ldots\approx 7.87$

## Exercise 2.4

1 Approximate the following squares to three decimal places using scientific calculators.

a $(4.25)^2$ c $(-4.3)^2$

b $\left(\frac{5}{3}\right)^2$

2 Approximate the following square roots to three decimal places using scientific calculators.

a $\sqrt{3.65}$ b $\sqrt{17}$ c $\sqrt{8.32}$

3 If the area of a square is 98 sq. units, then approximate the length of the side of the square to the nearest two decimal places. Compare your approximate values by using a scientific calculator.

### Use of Table of squares to Approximate Square Roots

In the previous section, you have seen how to find the square of a number from the Table of squares. Now let us see how to do the reverse process, that is, extracting a square root from a Table of squares and it is illustrated by using the following example.

## Example 2.12

Find $\sqrt{22.94}$ from the Table of squares.

### Solution

**Step 1:** From the body of the Table of squares, find 22.94.

**Step 2:** On the row containing the number 22.940, move horizontally to the left and read the number in the first column, which is 4.7, in this case.

**Step 3:** In the column containing 22.94 move vertically up ward and read the number in the first row, that is 9, in this case.

**Step 4:** Therefore, $\sqrt{22.94} \approx 4.79$

| x | 0 | 1 | 2 | 3 | 4 | | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.0 | | | | | | | | | | |
| .<br>.<br>. | | | | | | | | | | |
| 4.7 | | | | | | | | | | 22.940 |
| .<br>.<br>. | | | | | | | | | | |
| 9.9 | | | | | | | | | | |

Figure 2.5. Table of squares

## Example 2.13

Using the Table of squares, approximate square root $\sqrt{14.516}$ .

### Solution

**Step 1:** Find the number 14.516 on the body of the square table.

**Step 2:** On the row containing 14.516, move horizontally to the left and read the number on first column, which reads 3.8, in this case;

**Step 3:** On the column contacting 14.516 move vertically upward to first row and read the number, which reads 1 in this case;

Finally, the approximate value of $\sqrt{14.516}$ is 3.81

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.0 | 1.000 | 1.0[illegible]0 | 1.040 | 1.061 | 1.082 | 1.103 | 1.124 | 1.145 | 1.166 | 1.188 |
| 1.1 | 1.210 | 1.2[illegible]2 | 1.254 | 1.277 | 1.300 | 1.323 | 1.346 | 1.369 | 1.392 | 1.416 |
| | | | | | | | | | | |
| | | | | | | | | | | |
| 3.5 | 12.250 | 12[illegible]20 | 12.390 | 12.461 | 12.532 | 12.603 | 12.674 | 12.745 | 12.816 | 12.888 |
| 3.6 | 12.960 | 13[illegible]32 | 13.104 | 13.177 | 13.250 | 13.323 | 13.396 | 13.469 | 13.542 | 13.616 |
| 3.7 | 13.690 | 13[illegible]64 | 13.838 | 13.913 | 13.988 | 14.063 | 14.138 | 14.213 | 14.288 | 14.364 |
| 3.8 | [illegible] | 14.516 | 14.592 | 14.669 | 14.746 | 14.823 | 14.900 | 14.977 | 15.054 | 15.132 |
| 3.9 | 15.210 | 15.288 | 15.366 | 15.445 | 15.524 | 15.603 | 15.682 | 15.761 | 15.840 | 15.920 |
| 4.0 | 16.000 | 16.080 | 16.160 | 16.241 | 16.322 | 16.403 | 16.484 | 16.565 | 16.646 | 16.728 |

Figure 2.6. Table of squares

## Note

Sometimes you may not get the number that you want to find its square root in the given Table of squares. In such cases, take the square root of the number which is nearest to the given number.

### Example 2.14

Find $\sqrt{56.90}$ from the Table of squares.

**Solution**

The number 56.90 is not on the Table of squares, but it is between 56.85 and 57.00. The nearest of the two numbers to 56.90 is 56.85 and $\sqrt{56.85} \approx 7.54$ and hence $\sqrt{56.90} \approx 7.54$.

## Note

Table of squares are used to find the square roots of numbers whose square roots are from 1.00 to 9.99. The method to find the square root of numbers whose square roots are greater than 10.0 is illustrated below by using the following example.

### Example 2.15

Find $\sqrt{2841}$.

**Solution**

First express 2841 as $2841 = 28.41 \times 100$.

Then $\sqrt{2841} = \sqrt{28.41 \times 100} = \sqrt{28.41} \times \sqrt{100} = \sqrt{28.41} \times 10$, as $\sqrt{100} = 10$.

Then from the Table of squares, $\sqrt{28.41} \approx 5.33$.

Therefore, $\sqrt{2841} \approx 5.33 \times 10 = 53.30$.

## Note:

Note that values obtained from Table of squares are approximates.

### Exercise 2.5

1 Approximate the following squares using Table of squares.

a $(2.32)^2$ b $(9.2)^2$ c $(7.01)^2$ d $(0.32)^2$

2 Approximate the following square roots using Table of squares.

a $\sqrt{3.27}$ b $\sqrt{6}$ c $\sqrt{30.030}$ d $\sqrt{78.5}$

3 Write true or false for each of the following.

a $0.9 > (0.3)^2$

b $\sqrt{0.04} > 0.4$

c $\sqrt{0.01} < 0.1$

d $\sqrt{0.04} > 0.4$

4 Find each square root.

a $\sqrt{9}$

b $\sqrt{2.25}$

c $\sqrt{441}$

d $\sqrt{0.361}$

e $\sqrt{\frac{121}{225}}$

5 Arrange the following numbers in an increasing order: $\frac{1}{2}, \sqrt{0.01}, \sqrt{\frac{1}{2}}, 3, \sqrt{7}, \sqrt{10}$.

6 List all whole numbers whose squares are between 1 and 100.

7 If 69 tiles are arranged in a maximum square, how many tiles are left over?

8 Explain why −16 has no square root.

9 Using prime factorization find the square roots of the following products.

a $16 \times 6$ b $25 \times 49 \times 9$ c $12 \times 35 \times 63$

10 Find the square roots of the following numbers using the numerical table.

a 234

b 12321

c 0.099

d 4.356

11 Determine the lengths of sides of a square whose area is 4.63cm$^2$. (You may use Table of squares)

## 2.3 Cubes and Cube Roots

In the previous section, you have learned squaring a number and determining a number whose square is a given number, extracting a square root. In this section, you will learn how to multiply a number with itself three times, which is called cubing a number and also finding a number whose cube is given, extracting a cube root.

### 2.3.1 Cube of Rational Number

**Activity 2.5**

1 Find the volume of a cube of edge 4 cm.

2 How many cubes of edge 1 cm will make a cube of the following edge?

a 6 cm  b 7cm

3 Complete the following table.

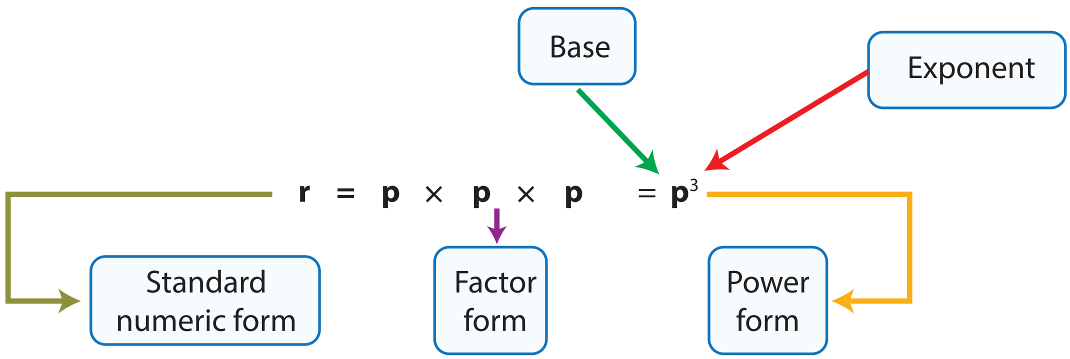
| y | −4 | −3 | −2 | −1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| $y \times y \times y$ | | −27 | | | | | | | |

In your responses in Activity 2.5, observe that, you can multiply a number with itself three times and obtain another number and the process is called cubing the cumber. The number obtained is called the cube of the given number.

### Definition 2.4

The process of multiplying a number by itself three times is called cubing the number. The product of $y$ by itself three times is called the cube of $y$, denoted by $y^3 = y \times y \times y$ and read as "$y$ cubed" or "$y$ to the power of 3".

The cube of a number is the value of the number to the power of 3. That is, if number $r$ is the cube of number $p$, then we write this relation as:




## Example 2.16

Evaluate the following cubes of numbers.

a $7^3$  b $9^3$  c $10^3$

**Solution**

By definition of a cube of a number, we have

a $7^3 = 7 \times 7 \times 7 = 7^2 \times 7 = 49 \times 7 = 343$.

b $9^3 = 9 \times 9 \times 9 = 9^2 \times 9 = 81 \times 9 = 729$.

c $10^3 = 10 \times 10 \times 10 = 10^2 \times 10 = 100 \times 10 = 1000$.

### Definition 2.5

A rational number $y$ is said to be a perfect cube if $y=x^3$, for some rational number $x$.

## Example 2.17

Show that each of the following numbers are perfect cubes.

a 64

b 125

c $\frac{8}{27}$

d 0.001

**Solution**

a $64=4^3$

b $125=5^3$

c $\frac{8}{27}=\left(\frac{2}{3}\right)^3$

d $0.001=(0.1)^3$

Therefore, 64,125, $\frac{8}{27}$ and 0.001 are all perfect cubes.

## Example 2.18

Show that 9 is not a perfect cube.

**Solution**

$9 = 3 \times 3$ and there is no whole number which is multiplied with itself three times and gives you 9.

Therefore, 9 is not a perfect cube.

## Note

You can use prime factorization to determine whether a given whole number is a perfect cube or not.

In the prime factorization of any number, if each factor appears three times, then the number is a perfect cube, otherwise the number is not a perfect cube.

## Example 2.19

Which of the following numbers are perfect cubes?

a 729 b 500 c 8000

**Solution**

a $729=(3\times3)\times(3\times3)\times(3\times3)$, that is, factors can be grouped in triples.

Therefore, 729 is a perfect cube and $729=(3\times3)^3=9^3$.

b $500 = (2 \times 2) \times (5 \times 5 \times 5)$. There are three 5's in the product, but only two 2's. Therefore, 500 is not a perfect cube.

c $8000 = (2 \times 2 \times 2) \times (2 \times 2 \times 2) \times (5 \times 5 \times 5)$. There are two three 2's and one three 5's in the product. Therefore, 8000 is a perfect cube and $8000 = (2\times2\times5)^3 = 20^3$

## Exercise 2.6

1 Find the cube of each of the following numbers.

a $\frac{1}{2}$ b 0.3 c $-\frac{4}{5}$ d $\frac{7}{3}$

2 Which of the following numbers are perfect cubes? Justify your answer.

a 343 d 9000 g 2025

b 400 e 15625 h 512000

c 3375 f 6859

3 Which of the following numbers in decimal form are perfect cubes? Justify your answer.

a 0.09 c 0.81 e 0.125

b 0.9 d 0.001

### 2.3.2 Cube Roots of Rational Numbers

In the previous discussion, you have seen how to find the cube of a number. In this part, you will see the reverse process of cubing a number, which is, finding a number whose cube is a given number.

## Activity 2.6

1 If the volume of a cube is 125 $cm^3$, then find the length of its edge.

2 For each of the following numbers, find a number whose cube is the given number.

a 0 c 27 e 1000

b 1 d 64

In your responses in Activity 2.6, you have been finding a number whose cube is a given number. It is reversing of cubing a number. The reverse process of finding the cube of a number is called extracting a cube root.

### Definition 2.6

If $y$ is the cube of a number $x$, that is $y = x^3$, then $x$ is called the cube root of $y$. This can be written symbolically as $x=\sqrt[3]{y}=\sqrt[3]{x^3}$

In the symbol $\sqrt[3]{y}$ , 3 is called the index, $\sqrt{\ }$ is the radical sign and y is the radicand.

## Example 2.20

Evaluate the following cube roots.

a $\sqrt[3]{-512}$

b $\sqrt[3]{\frac{1}{27}}$

c $\sqrt[3]{0.001}$

### Solution

a Since -512 is a perfect cube as it can be expressed as $(-8)^3=(-8)\times(-8)\times(-8)=-512$ then the cube root of -512 is -8. That is $\sqrt[3]{-512}=-8$

b $\left(\frac{1}{3}\right)^3=\frac{1}{3}\times\frac{1}{3}\times\frac{1}{3}=\frac{1}{27}$. Then the cube root of $\frac{1}{27}$ is $\frac{1}{3}$, i.e $\sqrt[3]{\frac{1}{27}}=\frac{1}{3}$.

c $(0.1)^3=0.1\times0.1\times0.1=0.001$. Thus the cube root of 0.001 is 0.1, i.e $\sqrt[3]{0.001}=0.1$.

## Exercise 2.7

1 Evaluate the following cube roots.

a $\sqrt[3]{0.125}$

b $\sqrt[3]{-1}$

c $\sqrt[3]{27{,}000}$

d $\sqrt[3]{\frac{1}{64}}$

2 If $\sqrt[3]{n}=5$, then find the possible value(s) of $n$.

3 When is the number $r=2\times\ 2\times\ 2\times\ 2\times\ 2\times\ldots\times\ 2$ a perfect cube? What can you say about number of 2'$s$?

4 Students in a class were asked to write "the cube root of sixty-four squared" in radical form. One student wrote $\left(\sqrt[3]{64}\right)^2$ and another student wrote $\sqrt[3]{(64)^2}$ .

a Simplify each expression.

b Who is correct? What can you say about the two expressions?

5 If the volume of a cubic box is 729 cubic units, then find the length of the edge of the box.

### 2.3.3 Approximations of Cubes and Cube Roots

As in the case of square and square roots, you can approximate cube and cube roots of numbers using scientific calculators and Tables of cubes. The methods are illustrated in the following examples.

**Example 2.21**

Approximate the cube of 2.16 by using

a a scientific calculator

b a Tables of cubes

**Solution**

a Type 2.16 on your scientific calculator and then press the sign $x^3$ . Then it will give you 10.077696 and rounding this number to three decimal point gives you 10.078.

Therefore, $(2.16)^3 \approx 10.078$.

b Consider a Table Cubes as in Figure 2.6;

**Step 1:** Find the row which starts with 2.1 under the column headed by x;

**Step 2:** move to the right until you get the column headed by 6 on the column of the table, as in Figure 2.7;

**Step 3:** Read the number at the intersection of the row in Step 1 and the column in Step 2, which is 10.078.

Therefore, $(2.16)^3 \approx 10.078$.

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.0 | 1.000 | 1.030 | 1.061 | 1.093 | 1.125 | 1.158 | 1.191 | 1.225 | 1.260 | 1.295 |
| 1.1 | 1.331 | 1.368 | 1.405 | 1.443 | 1.482 | 1.521 | 1.561 | 1.602 | 1.643 | 1.685 |
| 1.2 | 1.728 | 1.772 | 1.816 | 1.861 | 1.907 | 1.953 | 2.000 | 2.048 | 2.097 | 2.147 |
| 1.3 | 2.197 | 2.248 | 2.300 | 2.353 | 2.406 | 2.460 | 2.515 | 2.571 | 2.628 | 2.686 |
| 1.4 | 2.744 | 2.803 | 2.863 | 2.924 | 2.986 | 3.049 | 3.112 | 3.177 | 3.242 | 3.308 |
| 1.5 | 3.375 | 3.443 | 3.512 | 3.582 | 3.652 | 3.724 | 3.796 | 3.870 | 3.944 | 4.020 |
| 1.6 | 4.096 | 4.173 | 4.252 | 4.331 | 4.411 | 4.492 | 4.574 | 4.657 | 4.742 | 4.827 |
| 1.7 | 4.913 | 5.000 | 5.088 | 5.178 | 5.268 | 5.359 | 5.452 | 5.545 | 5.640 | 5.735 |
| 1.8 | 5.832 | 5.93 | 6.029 | 6.128 | 6.230 | 6.332 | 6.435 | 6.539 | 6.645 | 6.751 |
| 1.9 | 6.859 | 6.968 | 7.078 | 7.189 | 7.301 | 7.415 | 7.530 | 7.645 | 7.762 | 7.881 |
| 2.0 | 8.000 | 8.121 | 8.242 | 8.365 | 8.490 | 8.615 | 8.742 | 8.870 | 8.999 | 9.129 |
| 2.1 | 9.261 | 9.394 | 9.528 | 9.664 | 9.800 | 9.938 | 10.078 | 10.218 | 10.360 | 10.503 |
| 2.2 | 10.648 | 10.794 | 10.941 | 11.09 | 11.239 | 11.391 | 11.543 | 11.697 | 11.852 | 12.009 |

Figure 2.7. Table of Cubes

## Example 2.22

Approximate $\sqrt[3]{19.683}$ by using

a a scientific calculator

b a Tables of cubes

**Solution**

a Type 19.683 on your scientific calculator and then press the sign $\sqrt[3]{x}$ and it gives you 2.7.

Therefore, $\sqrt[3]{19.683} \approx 2.70$.

b Consider a Table of Cubes, as in Figure 2.8:

**Step 1**: Find the number 19.683 in the body of the Table of cubes;

**Step 2:** From the number move horizontally to the left on the row containing the number and read the number in the first column headed by x, the number is 2.1 in this case;

**Step 3:** From the number move vertically upward and read the number in the first row, which is 0, in this case.

Therefore, $\sqrt[3]{19.683} \approx 2.70$

| x | 0 | 1 | 2 |
|---|---|---|---|
| 1.0 | 1.000 | 1.030 | 1.061 |
| 1.1 | 1.331 | 1.368 | 1.405 |
| 1.2 | 1.728 | 1.772 | 1.816 |
| 1.3 | 2.197 | 2.248 | 2.300 |
| 1.4 | 2.744 | 2.803 | 2.863 |
| 1.5 | 3.375 | 3.443 | 3.512 |
| 1.6 | 4.096 | 4.173 | 4.252 |
| 1.7 | 4.913 | 5.000 | 5.088 |
| 1.8 | 5.832 | 5.93 | 6.029 |
| 1.9 | 6.859 | 6.968 | 7.078 |
| 2.0 | 8.000 | 8.121 | 8.242 |
| 2.1 | 9.261 | 9.394 | 9.528 |
| 2.2 | 10.648 | 10.794 | 10.941 |
| 2.3 | 12.167 | 12.326 | 12.487 |
| 2.4 | 13.824 | 13.998 | 14.172 |
| 2.5 | 15.625 | 15.813 | 16.003 |
| 2.6 | 17.576 | 17.78 | 17.985 |
| 2.7 | 19.683 | 19.903 | 20.124 |
| 2.8 | 21.952 | 22.188 | 22.426 |
| 2.9 | 24.389 | 24.642 | 24.897 |
| 3.0 | 27.000 | 27.271 | 27.544 |
| 3.1 | 29.791 | 30.080 | 30.371 |
| 3.2 | 32.768 | 33.076 | 33.386 |

Figure 2.1. Table of Cubes

## Exercise 2.8

1 Find the cube of the following numbers by using scientific calculators.

a 6.09 b 4.87 c 9.85 d 4.40

2 Find the cube of the following numbers by using Table of Cubes.

a 1.34 b 6 c 9.90 d 9.99

3 Find the cube root of the following numbers by using scientific calculators.

a 6.09 b 19.46 c 40.35 d 502.46

4 Find the cube roots of the following numbers by using Table of Cubes.

a 7.07 b 37.93 c 170.03 d 868.25

# Unit Summary

1 The square and cube of a natural number is also a natural number.

2 The square of an integer is also a whole number.

3 The square and cube of a rational number is also rational number.

4 A number with units digit in 2, 3, 7 and 8 is not a square number.

5 A number ending in an odd number of zeros is not square number.

6 The product of any number by itself gives a square number.

7 The process of finding squares of numbers and square roots are inverse process. That is, $b=a^2$ is equivalent to $\sqrt{b}=a,\ b\geq 0$ and $a>0$.

8 The process of finding cubes of numbers and cube roots are inverse process. That is, $b=a^3$ is equivalent to $\sqrt[3]{b}=a$ .

## Review Exercises

1 Give an example of

a A perfect square which is between 7 and 10.

b A perfect square between 0 and 1.

c A perfect cube between 0 and 1.

d Two distinct perfect square numbers whose sum is not a perfect square.

e Two distinct perfect square numbers whose sum is also perfect square.

2 Find the square and cube of each of the following numbers using table value.

a 8.45

b 2.03

c 8.99

d 0.76

3 Give an example to show that the sum of two perfect squares need not be a perfect square.

4 Show that the sum of the first 10 odd whole numbers 1 + 3 + 5 + ... + 19 is a perfect square.

5 Find the square roots and cube roots of the following numbers using Table of squares and Table of cubes.

a 11.24

b 94.82

c 413.5

d 909.9

6 Compute and compare the values of the following pairs of numbers.

a $\sqrt{4}\times\sqrt{25}$ and $\sqrt{4\times25}$

b $\sqrt[3]{8}\times\sqrt[3]{64}$ and $\sqrt[3]{8\times64}$

7 Are both 5 and -5 cube roots of $\sqrt[3]{125}$ ? Why?

8 What is the value of $-\sqrt[3]{125}$ ? What about $\sqrt[3]{-125}$ ?

9 Arrange the following rational numbers from smallest to largest.

a $\sqrt{15}$, $\sqrt[3]{17}$, 2, $\sqrt{103}$, $\sqrt[3]{124}$

b $\sqrt[3]{29}$, 5, $\sqrt{19}$, $\sqrt[3]{108}$, $\sqrt{105}$

c $\sqrt{27}$, 6, $\sqrt[3]{212}$, $\sqrt{101}$, $\sqrt[3]{101}$

d $\sqrt{12}$, 4.5, $\sqrt{41}$, 6.4, $\sqrt{55}$, 7

10 Given the following pattern of square numbers.

$1^2=1=1$

$2^2=4=1+3$

$3^2=9=1+3+5$

$4^2=16=1+3+5+7$

$\vdots$

What can you conclude about $1 + 3 + 5 + \cdots + (2n - 1)$ for a natural number n?

# Unit 3

# LINEAR EQUATIONS AND INEQUALITIES

## Learning outcomes:

**After completing this unit, you will be able to:**

- solve linear equations;
- solve linear inequalities;
- draw the graphs of linear equations;
- draw the graphs of linear inequalities;
- apply the concepts of linear equations and inequalities to solve real life problems.

## Key terms

- linear equations
- linear inequalities
- solution of linear equations
- solutions of linear inequalities
- equivalent equations
- equivalent inequalities
- properties of equality
- properties of inequality
- graph of equations
- graphs of inequalities

## Introduction

Linear equations and linear inequalities in one variable are commonly used in our daily activities. For example, supposeyour mother buys 100 Kg of teff by 4000 Birr and you want to determine the price of 1kg of teff. Let the price of 1kg of teff be t and you will have an equation $100 \times t = 4000$. When you solve for t, you will have the price of 1kg of teff. The expression $100 \times t = 4000$ is an example of a linear equation.

Suppose you have 2000 Birr and you want to buy reference books with a price of 120 Birr each, Let b be the number of books that you can buy with 2000 Birr. Then to determine the maximum number of books that you can buy with 2000 Birr, you need to solve the inequality $120 \times b \leq 2000$ and this is an example of a linear inequality.

In this unit, you will learn how to solve linear equations and linear inequalities, You will also learn how to draw linear equations and linear inequalities in the Cartesian Coordinate Plane and some applications of linear equations and line inequalities in real-life.

## 3.1 Solving Linear Equations

### Activity 3.1

1 If the sum of three consecutive integers is 18, then find the numbers.

2 A man's age is 5 times the age of his daughter and the sum of the ages of the man and his daughter is 48. Find the ages of both the man and his daughter.

In your responses in Activity 3.1, observe that, in the process of the solutions, linear equations are involved.

### Definition 3.1

Any expression that can be reduced to the form $ax = b$, where $a$ and $b$ are numbers and $a \neq 0$, is called a linear equation in one variable, $x$

### Example 3.1

Which one of the following are linear equations in one variable?

a $3x=2$

b $2x + 5 = 12$

c $5x – 3 = 7 – 2x$

d $\frac{1}{x} - 3 = 0$

e $\sqrt{x} + 2 = 5$

f $x - y = 1$

**Solution**

a $3x=2$ is a linear equation in one variable x.

b $2x + 5 = 12$ implies $2x + 5 – 5 = 12 –5$.

Thus, $2x = 7$, a linear equation in one variable x.

Therefore, $2x + 5 = 12$ is a linear equation in one variable x.

c $5x - 3 = 7 - 2x$ implies $5x - 3 = 7 - 2x \Rightarrow 7x = 10$ is a linear equation in one variable x

So, $5x - 3 = 7 - 2x$ is a linear equation in one variable

d $\frac{1}{x} - 3 = 0$ is not a linear equation, since it involves $\frac{1}{x}$.

e $\sqrt{x} + 2 = 5$ is not a linear equation, because it involves $\sqrt{x}$.

f $x - y = 1$ is not a linear equation in one variable, since it involves two variables $x$ and $y$.

### Solutions of Linear Equations

Given a linear equation $ax = b$:

i a number $c$ such that $a \times c = b$ is called a solution to the given equation;

ii the set of all numbers that are solutions of the given linear equation is called the solution set of the equation;

iii solving a linear equation means, finding all the numbers that satisfy the given equation.

## Example 3.2

Which of the following numbers is a solution to the linear equation 3x =12?

a 1 b 2 c 3 d 4

**Solution**

a 3 x 1 = 12

4 = 12 (False)

So, 1 is not a solution

b 3 x 2 = 12

6 = 12 (False)

Therefore, 2 is not a solution

c 3 x 3 = 12

9 = 12 (False)

So, 3 is not a solution

d 3 x 4 = 12

12 = 12

Therefore, 4 is a solution

In solving linear equations, the following rules of addition, subtraction, multiplication and division of numbers are important and used very frequently.

### Rules of Equalities

Let $a$, $b$ and $c$ be rational numbers.

**Rule I(Addition Rule):** Adding the same number on both sides of an equation does not change the solution of the equation. That is, if $a = b$, then $a + c = b + c$.

**Rule II (Subtraction Rule):** Subtracting the same number from both sides of an equation does not change the solution of the equation. That is, if $a = b$, then $a - c = b - c$.

**Rule III (Multiplication Rule):** Multiplying both sides of an equation by the same nonzero number does not change the solution of the equation. That is, if $a = b$, then $a \times c = b \times c$.

**Rule IV (Division Rule):** Dividing both sides of an equation by the same nonzero number does not change the solution of the equation. That is, if $a = b$ and $c \neq 0$, then $\frac{a}{c} = \frac{b}{c}$.

Commutative properties of both addition and multiplication, associative properties of addition and multiplication and distributive properies of multiplication over additionare frequently used in solving linear equations.

Let $a$, $b$ and $c$ be rational numbers. Then

i $a + b = b + a$ (Commutative property of addition)

ii $ab = ba$ (Commutative property of multiplication)

iii $(a+b)+c = a+(b+c)$ (Associative property of addition)

iv $(ab)c = a(bc)$ (Associative property of multiplication)

v $a(b+c) = ab+ac$ (Distributive of multiplication over addition)

vi $(a+b)c = ac+bc$ (Distributive of multiplication over addition)

vii $a(b-c) = ab-ac$ (Distributive of multiplication over subtraction)

viii $(a-b)c = ac-bc$ (Distributive of multiplication over subtraction)

## Example 3.3

Solve each of the following linear equations.

a $2x = 10$

b $3x - 5 = 4$

c $4(x - 1) + 3(x + 2) = 5(x - 4)$

d $\frac{1}{3}(x + 1) - \frac{1}{6}(x - 3) + \frac{1}{4}x = 10$

**Solution**

a $2x=10$

$\frac{2x}{2} = \frac{10}{2}$ (divided both sides by 2)

$x = 5$

Check: $2(5) = 10$

$10 = 10$

So, $x = 5$ is the solution of the given equation b $3x - 5 = 4$

$3x - 5 + 5 = 4 + 5$ (add 5 on both sides)

$3x = 9$

$\frac{3x}{3} = \frac{9}{3}$ (divided both sides by 3)

$x = 3$

Check: $3(3) - 5 = 4$

$9 - 5 = 4$

$4 = 4$ True

Thus, $x = 3$ is the solution of the given equation

c $4(x - 1) + 3(x + 2) = 5(x - 4)$

$4x - 4 + 3x + 6 = 5x - 20$ (Distributive property)

$7x + 2 = 5x - 20$ (Commutative property and adding like terms)

$7x + 2 - 2 = 5x - 20 - 2$

$7x = 5x - 22$

$7x - 5x = 5x - 5x - 22$ (Subtracting 2 from both sides)

$2x = -22$

$\frac{2x}{2} = \frac{-22}{2}$ (Dividing both sides by $2$)

$x = -11$

Check: $4(-11 - 1) + 3(-11 + 2) = 5(-11 - 4)$.

$4(-12) + 3 - (-9) = 5(-15)$

$-48 - 27 = -75$

$-75 = -75$ (True)

Therefore, $x = -11$ is the solution of the given equation.

d $\frac{1}{3}(x + 1) - \frac{1}{6}(x - 3) + \frac{1}{4}x = 10$

$\frac{12}{3}(x + 1) - \frac{12}{6}(x - 3) + \frac{12}{4}x = 10 \times 12$ (Multiply both sides by 12)

$$4(x+1) - 2(x-3) + 3(x) = 120$$

$$4x + 4 - 2x + 6 + 3x = 120 \quad \text{(Distributive property)}$$

$$5x = 110 \quad \text{(Collecting like terms)}$$

$$\frac{5x}{5} = \frac{110}{5} \quad \text{(Dividing both sides by 5)}$$

Hence, $x = 22$

Check: $\frac{1}{3}(22+1) - \frac{1}{6}(22-3) + \frac{1}{4}(22) = 10$

$$\frac{23}{3} - \frac{19}{6} + \frac{22}{4} = 10$$

$$\frac{4(23) - 2(19) + 3(22)}{12} = 10$$

$$\frac{92 - 38 + 66}{12} = 10$$

$$\frac{120}{12} = 10$$

$$10 = 10 \text{ (True)}$$

Therefore, $x = 22$ is the solution of the given equation.

## Solving Word Problems using Linear Equations

To solve a word problem involving linear equations, first the problem has to be experessed as a mathematical equation and then solve the mathematical equation.

The following are steps recommended to solve word problems.

**Step 1:** Read the word problem carefully and identify the unknowns whose values are needed in the answer of the question.

**Step 2:** Represent each of the unknowns by letters (variables).

**Step 3:** Using the information given in the word problem, express the verbal statements by appropriate mathematical expressions or equations in terms of the corresponding variables.

If more than one variable are involved, look for additional information that can help you to form additional equations that represent the relation between the variables. Using these equations and appropriate substitutions, form equation that contains only one variable.

**Step 4:** Solve the equation.

## Example 3.4

If the sum of two consecutive even integers is 206, then find the numbers.

### Solution

The unknowns in this questions are two consecutive even integers.

Let one of the integers be $x$. Then the next even integer is $x + 2$.

$x + x + 2 = 206$ (the equation representing the given information)

$2x = 204$ (adding like terms)

$\frac{2x}{2} = \frac{204}{2}$ (divided both sides by 2)

$x = 102$ is the first number.

The second number is $x + 2 = 102 + 2 = 104$. Therefore the two consecutive even integers whos sum is 206 are 102 and 104.

## Example 3.5

If the sum of three consecutive natural numbers is increased by 2, the result is 50. Find the numbers.

### Solution

The unknowns in this problem are three consecutive natural numbers.

Let us represent the numbers by variables. Let

x = the first (smallest) of the three numbers;

y = The second (next to x) number; and

z = The third (largest) number (next to y).

Then, expression 'the sum of the three numbers' is expressed mathematicaly as x + y + z; and 'this sum is increased by 2' is expressed mathematically as: (x + y + z ) + 2.

Then the given statement is x + y + z + 2 = 50.

This equation contains three variables and change the equation into an equation with only one variable.

The numbers are consecutive, that is: $y = x + 1$ and $z = y + 1 = x + 1 + 1 = x + 2$.

Now substitute these in the equation x + y + z + 2 = 50.

That is, $x + x + 1 + x + 2 + 2 = 50$.

$3x + 5 = 50$ (adding like variables)

$3x + 5 - 5 = 50 - 5$ (subtracting 5 from both sides)

$3x = 45$

$\frac{3x}{3} = \frac{45}{3}$ (dividing both sides by 3)

$x = 15$.

Then solving the three variables: $x = 15$, $y = x + 1 = 15 + 1 = 16$ and $z = y + 1 = 16 + 1 = 17$.

Therefore, the required three consecutive numbers are 15, 16 and 17.

## Equivalent Equations

### Activity 3.2

Solve each of the following pairs of equations

a $2x - 5 = 11$ and $3x = 24$

b $x + 4 = 5$ and $3 - x = -2$

From your response in Activity 3.2, observe that the pair of equation in (a) have the same solution. Such equations are called equivalent equation

### Definition 3.2

Two linear equations are said to be equivalent equations, if they have the same solution set.

### Example 3.6

Show that the linear equations $3x + 4 = 10$ and $\frac{1}{2}(3x - 2) = 2$ are equivalent.

**Solution**

First solve the two equations.

$3x + 4 = 10$

$3x + 4 - 4 = 10 - 4$

$3x = 6$

$\frac{1}{3}(3x) = \frac{6}{2}$

$x = 2$

Therefore, $x = 2$ is the solution.

$\frac{1}{2}(3x - 2) = 2$

$2 \times \left(\frac{1}{2}(3x - 2)\right) = 2 \times 2$

$3x - 2 = 4$

$3x - 2 + 2 = 4 + 2$

$3x = 6$

$\frac{1}{3}(3x) = \frac{6}{2}$

$x = 2$

Therefore, $x = 2$ is the solution.

The two equations have the same solution, $x = 2$ and hence they are equivalent equations.

## Example 3.7

Determine whether the following equations are equivalent to $3x + 15 = 0$ or not.

a $2x + 3 = -7$

b $2(x-1) = 18$

c $3x + 1 = -14$

### Solution

First find the solution set of the given equation.

$3x + 15 - 15 = 0 - 15$ (subtracting 15 from both sides of the given equation)

$3x = -15$

$\frac{3x}{3} = \frac{-15}{3}$ (dividing both sides of the given equation by 3)

Therefore, $x = -5$ is the solution of the given equation.

Now, find the solutions of the equations in (a), (b) and (c) and compare each solution with $x = -5$.

a $2x + 3 = -7$

$2x + 3 - 3 = -7 - 3$

$2x = -10$

$\frac{2x}{2} = \frac{-10}{2}$

$x = -5$

This implies, $x = -5$ is the solution of the given equation.

b $2(x-1) = 18$

$2x - 2 = -18$

$2x - 2 + 2 = -18 + 2$

$2x = -16$

$\frac{2x}{2} = \frac{-16}{2}$

$x = -8$

So, $x = -8$ is the solution of the given equation

c $3x + 1 = -14$

$3x + 1 - 1 = -14 - 1$

$3x = -15$

$\frac{3x}{3} = \frac{-15}{3}$

$x = -5$

This implies, $x = -5$ is the solution of the given equation.

Thus, $x = -5$ is the solution for the equations in (a) and (c), which is the same as the solution of the equation $3x + 15 = 0$.

Thus, the equations in (a) and (c) are equivalent to the given equation $3x + 15 = 0$.

The equation in (b) is not equivalent to the equation $3x + 15 = 0$, because their solutions are not the same.

**Remark:**

1 An equation may be satisfied by every rational number and then the solution set of the given equation is the set of all rational numbers .

2 There are some equations which cannot be satisfied by any rational number. The solution set of such equations is empty set, written as {} or ∅.

## Example 3.8

Solve the equation $2x + 3(x - 5) + 4 = 5x - 11$.

**Solution**

$2x + 3(x - 5) + 4 = 5x - 11$ (The given equation)

$2x + 3x - 15 + 4 = 5x - 11$ (by distributive property of multiplication over subtraction)

$5x - 11 = 5x - 11$ (Combining like terms)

$-11 = -11$ (Subtracting x from both sides)

The last statement is always true whatever the value of x you take.

Thus, the solution set is the set of all numbers in the specified value of x, set of all rational numbers in this case.

## Example 3.9

Solve the equation $3+x = x$.

**Solution**

$3+x = x$

$3+x - x = x - x$ (Subtracting x from both sides of the equation)

$3=0$ (which is false)

This implies, there is no rational number that satisfies the given equation.

Hence, the solution set of the given equation is empty set ({} or ∅).

## Exercise 3.1

1 Solve each of the following linear equations.

a $3 + 2(2x + 1) = 11$

b $5(x + 3) - 2(2x + 5) = -5$

c $-(x + 2) - 3(x - 2) = 4(2x - 1)$

d $\frac{5}{6}x + \frac{2}{5} = -\frac{1}{6}x - \frac{5}{3}$

e $\frac{3}{7}x - \frac{1}{4} = -\frac{4}{7}x + \frac{5}{4}$

f $0.45 - \frac{2}{13}x = \frac{2}{3}x - 0.5$

2 If the sum of two consecutive odd integers is 300, then find the numbers.

3 The perimeter of a rectangular garden is 144m. If the length of the garden is 14m more than its width, then find the dimensions of the rectangle.

4 Suppose the current temperature is 54°F and it is expected to rise 0.2°F in every hour for the next several hours. In how many hours will the temperature be 78°F?

5 Hana's father is 4 times as old as Hana and Hana's mother is 7 years younger than her father. If their three ages are add up to 101 years, how old is Hana?

6 Which of the following equations are equivalent to the equation $\frac{1}{2}(3x - 2) = 5$?

a $3x - 2 = 10$ b $2x + 1 = 7$ c $3x - 12 = 0$ d $x - 8 = 0$

7 What is the solution set of equation $-3(2x - 1) + 2(3x + 4) = 11$ in the set of rational numbers?

8 Find the solution set of the equation $2(3x - 5) = 3(2x + 4)$ in the set of rational numbers.

## 3.2 Solving Linear Inequalities

In the previous section, you have learnt how to solve linear equations. In this section, you will learn how to solve linear inequalities and applications of linear inequalities in solving real-life problems.

### Activity 3.3

A farmer wants to fence his farm land with fencing materials of length as shown in Figure 3.1.

a Find the value of $x$ for which the fencing material is exactly 370m.

b Find two possible values of $x$ for which the fencing material is less than 370m.

c Find two possible values of $x$ for which the fencing material is greater than 370m.

d Express the above statements mathematically and discuss their differences. What make (b) and (c) differ from (a)?

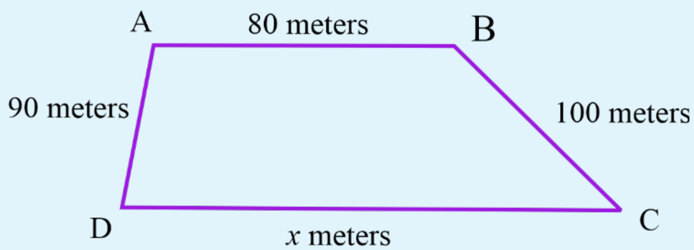



Figure 3.1

From your responses in Activity 3.3, observe that the perimeter of the fence is the sum of the lengths of the four sides of the figure and the sum of the lengths of the four sides may be less than or greater than the fencing material that the former has.

### Definition 3.3

Let $a$ and $b$ are numbers such that $a \neq 0$. An inequality in one variable, say $x$, that can be reduced to any one of the form $ax + b < 0$ or $ax + b \leq 0$ or $ax + b > 0$ or $ax + b \geq 0$ is called a linear inequality.

## Example 3.10

Which of the following are linear inequalities in one variable?

a $2x < 10$

b $3x + 6 > 8$

c $-12x + 9 \geq 4$

d $-2x + 23 = 11$

e $\frac{2}{5}x = 13$

f $\frac{2}{3}x - 23 = 32$

g $\frac{3}{4}x - \frac{4}{7} \leq 0$

h $-2.5x - 19 = 0$

### Solution

The expressions in a, b, c and g are linear inequalities and the expressions in d, e, f and h are not linear inequalities, but they are linear equations.

### Definition 3.4

The solution set for an inequality is the set of all numbers that satisfy the inequality, that is, the set of all the numbers that make the statement of the inequality true.

### Notation

Given a rational number c:

1. $\{x: x < c\}$ or $\{x \mid x < c\}$ read as " the set of $x$ in the set of rational numbers **such that** $x$ is less than $c$. This notation represents the set (collection) of **all rational numbers** each of which is **less than** $c$.

2. $\{x : x \leq c\}$ or $\{x|x \leq c\}$ read as " the set of $x$ in the set of rational numbers **such that** $x$ is **less than or equal to** $c$. This notation represents the set (collection) of **all rational numbers** each of which is **less than or equal to** $c$

3. $\{x : x > c\}$ or $\{x|x > c\}$ read as " the set of $x$ in the set of rational numbers **such that** $x$ is greater than $c$. This notation represents the set (collection) of **all rational numbers** each of which is **greater than** $c$.

4. $\{x : x \geq c\}$ or $\{x|x \geq c\}$ read as " the set of $x$ in the set of rational numbers **such that** $x$ is greater than or equal to $c$. This notation represents the set (collection) of **all rational numbers** each of which is **greater than or equal to** $c$.

The following Rules of inequality are very important to solve linear inequalities.

## Rules of inequalities

Let $a$, $b$ and $c$ be rational numbers.

**Rule 1:** If the same number is added or subtracted from both sides of an inequality, then the direction of the inequality sign is unchanged. That is

- If $a < b$, then $a + c < b + c$, and
- If $a < b$, then $a - c < b - c$

**Rule 2:** If both sides of an inequality are multiplied or divided by the same positive number, then the direction of the inequality sign is unchanged. That is

- If $a < b$ and $c > 0$, then $ac < bc$, and
- If $a < b$ and $c > 0$, then $\frac{a}{c} < \frac{b}{c}$

**Rule 3:** If both sides of an inequality are multiplied or divided by the same negative number, then the direction of the inequality sign is reversed. That is

- If $a < b$ and $c < 0$, then $ac > bc$, and
- If $a < b$ and $c < 0$, then $\frac{a}{c} > \frac{b}{c}$

The above rules hold true if "<" is replaced by ">" or "≤" or "≥".

### Example 3.11

Simplify the inequality $\frac{1}{2}(2x + 3) \leq \frac{5}{4}x$ to one of the forms $x \leq a$ or $x \geq a$

#### Solution

$\frac{1}{2}(2x + 3) \leq \frac{5}{4}x$

$2(2x + 3) \leq 5x$ (Multiply both sides by 4)

$4x + 6 \leq 5x$ (Distributive property of multiplication over addition)

$-x + 6 \leq 0$ (Subtract $5x$ from each side)

$-x \leq -6$ (Subtract 6 from each side)

$x \geq 6$ (Divide both sides by $-1$)

## Exercise 3.2

1. Write a mathematical equation or inequality for each of the following statements and identify the statements that represent linear inequalities.

   a. A number exceeds its double.

   b. The sum of two numbers is at least 10.

   c. The sum of a number and its successor is even.

   d. The difference of two integers is non-negative.

   e. A student scored above average in mathematics examination.

2. Simplify the following inequalities in one of the forms $x > a$ or $x < a$ or $x \leq a$ or $x \geq a$

   a. $2x - 13 > 43$

   b. $x - \frac{2}{9}x - \frac{7}{6} \leq 0$

   c. $-4x + 41 < 1$

   d. $-x - \frac{9}{2} \geq \frac{11}{5}$

### Definition 3.5

The set of all the value(s) of the variable in a specified set of numbers that satisfy the given inequality is called solution set or truth set of the inequality.

For rational numbers $a$ and $b$, consider the linear inequality $ax + b > 0$.

There are two cases to consider to solve the given linear inequality.

**Case 1:** If $a > 0$, then you can solve the inequality $ax + b > 0$ as follows.

$ax + b > 0$ (Given inequality)

$ax + b - b > 0 - b$ (Subtract $b$ from both sides )

$ax > -b$

$\frac{a}{a}x > \frac{-b}{a}$ (Divide both sides by $a$ )

$\boldsymbol{x > \frac{-b}{a}}$

Therefore, the solution set of the inequality $ax + b > 0$, when $a>0$ is all rational numbers greater than $-\frac{b}{a}$. This set is given by $\left\{x \in \mathbb{Q} \mid x > -\frac{b}{a}\right\}$

**Case 2:** If $a < 0$, then you can solve the inequality $ax + b > 0$ as follows

$ax + b > 0$

$ax + b - b > 0 - b$ (subtract b from both sides)

$ax > -b$

$\frac{ax}{a} < \frac{-b}{a}$ (divide both sides by a negative rational number $a$ and the inequality sign is reversed)

$x < \frac{-b}{a}$

Therefore, the solution set of the inequality $ax + b > 0$, when $a < 0$ is the set all rational numbers less than $-\frac{b}{a}$. This can also be expressed as: $\left\{x \in \mathbb{Q} \mid x < \frac{-b}{a}\right\}$.

## Note

The above rules hold true for "<" , "≤" and "≥" also.

1 The solution set of the inequality $ax + b \geq 0$ is:

i $\left\{x \in \mathbb{Q} \mid x \geq \frac{-b}{a}\right\}$ if $a > 0$;

ii $\left\{x \in \mathbb{Q} \mid x \leq \frac{-b}{a}\right\}$ if $a < 0$;

2 The solution set of the inequality $ax + b < 0$ is:

i $\left\{x \in \mathbb{Q} \mid x < \frac{-b}{a}\right\}$ if $a > 0$;

ii $\left\{x \in \mathbb{Q} \mid x > \frac{-b}{a}\right\}$ if $a < 0$;

3 The solution set of the inequality $ax + b \leq 0$ is:

i $\left\{x \in \mathbb{Q} \mid x \leq \frac{-b}{a}\right\}$ if $a > 0$;

ii $\left\{x \in \mathbb{Q} \mid x \geq \frac{-b}{a}\right\}$ if $a < 0$;

## Example 3.12

Solve each of the following linear inequalities.

a $2x < 10$

b $7x + 15 \leq 4x - 9$

c $-4x - 6 < \frac{1}{2}(28 - 2x)$

d $\frac{1}{2}(2x + 3) \leq \frac{5}{4}x$

### Solution

a $2x < 10$ (the given inequality)

$\frac{2x}{2} < \frac{10}{2}$ (dividing both sides by 2)

$x < 5$

Therefore, the solution set of the given inequality is $\{x \in \mathbb{Q} \mid x < 5\}$.

b $7x + 15 \leq 4x - 9$ (the given inequality)

$7x + 15 + 9 \leq 4x - 9 + 9$ (add 9 on both sides)

$7x + 24 - 4x \leq 4x - 4x$ (subtract $4x$ on both sides)

$3x + 24 \leq 0$

$3x \leq -24$ (subtracting 24 from both sides)

$\frac{3x}{3} \leq \frac{-24}{3}$ (divide both sides by 3)

$x \leq -8$

Therefore, the solution set of the given inequality is $\{x \in \mathbb{Q} \mid x \leq -8\}$

c $-4x - 6 < \frac{1}{2}(28 - 2x)$ (the given inequality)

$-4x - 6 < 14 - x$ (by distributive property of multiplication over addition)

$-4x < 14 - x + 6$ (add 6 on both sides)

$-4x + x < 14 + 6$ (add x from both sides)

$-3x < 20$

$\frac{-3x}{-3} > \frac{20}{-3}$ (divide both sides by-3 reverse the sign of the inequality).

$x > \frac{-20}{3}$

Therefore, the solution set of the given inequality is $\left\{x \in \mathbb{Q} \mid x > \frac{-20}{3}\right\}$.

d $\frac{1}{2}(2x+3) \leq \frac{5}{4}x$ (the given inequality)

$2(2x+3) \leq 5x$ (multiply both sides by 4 )

$4x + 6 \leq 5x$ (by distributive property of multiplication over addition)

$-x + 6 \leq 0$ (by subtracting $5x$ from both sides)

$-x \leq -6$ (by subtracting $6$ from both sides)

$x \geq 6$ (multiplying both sides by -1 reverse the inequality sign)

Therefore, the solution set of the given inequality is $\{x \in \mathbb{Q} | \ x \geq 6\}$.

## Example 3.13

In Figure 3.2, find all possible values of $x$ for which the perimeter of the polygon is not greater than 62km.

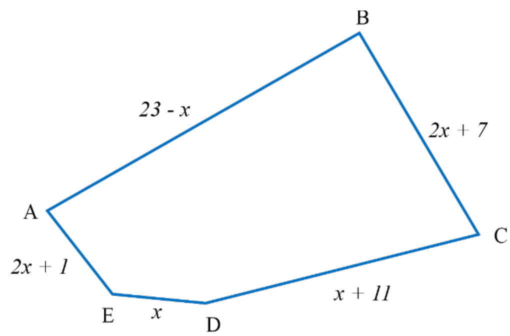


Figure 3.2

### Solution

Let $P$ be the perimeter of the polygon $ABCDE$. Then its perimeter is given by:

$$P = AB + BC + CD + DE + EA$$

$$P = (23 - x) + (2x + 7) + (x + 11) + x + (2x + 1)$$

$$P = 23 - x + 2x + 7 + x + 11 + x + 2x + 1$$

$$P = 42 + 5x$$

But the perimeter of the polygon is not greater than 62km.

That is $P \leq 62\,km$

$$42 + 5x \leq 62$$

$$42 - 42 + 5x \leq 62 - 42 \quad \text{(subtracting 42 from both sides)}$$

$$5x \leq 20$$

$$\frac{5x}{5} \leq \frac{20}{5}$$

$$x \leq 4$$

Therefore, the value of $x$ that makes the perimeter is not grater than 62km is greater 0 and less than or equal to 4. That is $\{x \in \mathbb{Q} : 0 < x \leq 4\}$. Note that 0 is not solution.

## Example 3.14

A farmer wants to plant more than 400 seedlings of mango, avocado and lemon in the coming summer. For this purpose, he prepared 176 Mango and 142 Avocado seedlings. Find the number of lemon seedlings that the farmer has to plant to achieve his goal.

### Solution

Let $x$ be the number of lemon seedlings that the farmer has to plant. Since the total numbers of seedlings that the farmer wants to plant is more than 400, you have

$$176 + 142 + x > 400$$

$$318 + x > 400 \quad \text{(Adding like terms)}$$

$$318 - 318 + x > 400 - 318 \quad \text{(Subtracting 318 from both sides)}$$

$$x > 82$$

Therefore, the farmer has to prepare more than 82 lemon seedlings to achieve his plan.

## Exercise 3.3

1 Simplify each the following inequalities into one of the forms $x > a$, $x < a$, $x \leq a$ or $x \geq a$.

a $2x - 13 > 43$

b $-4x + 41 < 1$

c $x - \frac{2}{9}x - \frac{7}{6} \leq 0$

d $-x - \frac{9}{2} \geq \frac{11}{5}$

2 Solve each of the following linear inequalities.

a $\frac{x}{4} + 5 \leq x + 4$

b $8x - 5 \geq 13 - x$

c $2(x + 3) < 3(x + 1)$

d $\frac{1}{4}x + 7 > \frac{1}{3}x - 2$

e $9 + \frac{1}{3}x \geq 4 - \frac{1}{2}x$

f $\frac{1}{2}(2x + 3) > 0$

3 A grade eight student has studied hard to score an average mark of over 96 in three subjects Mathematics, English and Physics. She scored 93 in Mathematics and 97 in Physics. How many marks should she score in English to achieve her goal?

## Equivalent Inequalities

### Activity 3.4

Solve and compare the solutions of each of the following pairs of linear inequalities.

a $2x - 10 \leq 0$ and $x \leq 5$  b $-4x - 6 \geq 0$ and $8x \geq 12$

From your responses in Activity 3.4, observe that the first pair of linear inequalities have the same solutions. Such linear inequatities are called equivalent inequalities.

### Definition 3.6

Two linear inequalities are said to be equivalent if they have the same solution set.

### Example 3.15

Which of the following pairs of linear inequalities are equivalent?

a $x + 15 \leq -9$ and $x + 3 \leq -21$  b $-3x - 11 > 7$ and $2x - 2 > -6$

**Solution**

a First let us solve both linear inequalities.

| | |
|---|---|
| $x + 15 \leq -9$ | $x + 3 \leq -21$ |
| $x \leq -9 - 15$ | $x \leq -21 - 3$ |
| $x \leq -24$ | $x \leq -24$ |

The solution set of both inequalities is $\{x \in \mathbb{Q} : x \leq -24\}$. That is the linear inequalities x + 12  –9 and  x + 3 –21 have the same solution set. Hence the two inequalities are equivalent.

b First let us solve both linear inequalities.

| | |
|---|---|
| $-3x - 11 > 7$ | $2x - 2 > -6$ |
| $-3x > 7 + 11$ | $2x > -6 + 2$ |
| $\frac{-3x}{-3} < \frac{18}{-3}$ | $2x > -4$ |
| $x < -6$ | $x > -2$ |

Thus, the solution of the inequality $-3x - 11 > 7$ is $\{x \in \mathbb{Q} : x < -6\}$ and the solution of the inequality $2x - 2 > -6$ is $\{x \in \mathbb{Q} : x > -2\}$.

This implies, the two inequalities do not have the same solution set.

Therefore, $-3x - 11 > 7$ and $2x - 2 > -6$ are not equivalent inequalities

## Example 3.16

Give at least three linear inequalities that are equivalent to the linear inequality:

$$4(2x - 3) > 3(2x + 4) - 12$$

### Solution

a $8x - 1\ 2 > 6x$ is equivalent to $4(2x - 3) > 3(2x + 4) - 12$. The linear inequality $8x - 12 > 6x$ is obtained by applying distributive property of multiplication over addition and then simplification.

b $2x > 12$ is equivalent to $4(2x - 3) > 3(2x + 4) - 12$, because $2x > 12$ is obtained by adding 12 and then subtracting 6x from both sides of the inequality $8x - 12 > 6x$.

c $x > 6$ is equivalent to $4(2x - 3) > 3(2x + 4) - 12$, because $x > 6$ is obtained by dividing both sides of $2x > 12$ by 2.

Therefore, the inequalities $8x - 12 > 6x$, $2x > 12$ and $x > 6$ are all equivalent to the inequality $4(2x - 3) > 3(2x + 4) - 12$.

## Exercise 3.4

1 Identify whether the following pairs of linear inequalities are equivalent or not.

a $2x - 13 > 43$ and $x > 28$

b $x - \frac{2}{9}x - \frac{7}{6} \leq 0$ and $x \leq \frac{3}{2}$

c $-x - \frac{9}{2} \geq \frac{11}{5}$ and $x \leq -\frac{76}{10}$

d $-4x + 41 < 1$ and $> x10$

2 Give at least three linear inequalities which are equivalent to linear inequality $2x - \frac{1}{4}x \leq \frac{5}{6}x$.

3 A man planned to walk an average of 50 kilometers in a week for four consecutive weeks. He walked 56 kilometers in the first week, 52 kilometers in the second week and 38 kilometers in the third week.

a Formulate a mathematical expression for problem.

b How many kilometers should he walk on the fourth week to meet his plan?

c How many kilometers should he walk in the fourth week to walk more than his plan?

## Domain of a Variable

### Activity 3.5

1 From the following set of numbers $\left\{2, 4, -3, \frac{4}{5}, 5, -8, \frac{3}{2}, -0.5, 0, 1. -6, \frac{11}{3}, 3.5\right\}$

a find the natural numbers that are less than 6;

b find the integers that are between –6 and 6

2 Given the inequality $x < 6$,

a find all the natural numbers that satisfy the given linear inequality;

b find some rational numbers that satisfy the given linear inequality.

From your responses in Activity 3.5, observe that, the solution set of a given inequality may vary based on the given set in which you can choose your solutions.

### Definition 3.7

The set of numbers from which the values of the variable must be chosen so that the given inequality is meaningful is called the domain of the variable.

### Example 3.17

Find the solution set of $-7x - 3(4 - 2x) > 0$ in the following domains.

a $\mathbb{W}$ b $\mathbb{Z}$ c $\mathbb{Q}$

**Solution**

First solve the inequality.

$-7x - 3(4 - 2x) > 0$

$-7x + 6x - 12 > 0$ (by distributive property of multiplication over addition)

$-x - 12 > 0$ (by adding like terms)

$-x > 12$ (by adding 12 on both sides)

$x < -12$ (by multiplying both sides by -1 and the inequality sign is reversed)

a Since there are no whole numbers less than –12, the inequality $-7x - 3(4 - 2x) > 0$ has no solution in the set of whole numbers.

b The integers that are less than -12 are $\ldots, -16, -15, -14, -13$. Therefore, the solution set of the given inequality in the set of integers is: $\{\cdots, -16, -15, -14, -13\}$

c The solution set of the inequality in the domain $\mathbb{Q}$ is: $\{x \in \mathbb{Q} : x < -12\}$

## Example 3.18

Find the solution set of $2(x+3) \leq \frac{3}{4}(x-4)+5x$ in the set of positive rational numbers.

**Solution**

$$2(x+3) \leq \frac{3}{4}(x-4)+5x$$

$$2x+6 \leq \frac{3}{4}x-3+5x$$

$$-3x+9 \leq \frac{3}{4}x$$

$$-12x+36 \leq 3x$$

$$-15x+36 \leq 0$$

$$-15x \leq -36$$

$$x \geq \frac{12}{5}$$

Therefore, the solution set of $2(x+3) \leq \frac{3}{4}(x-4)+5x$ in $\mathbb{Q}^+$ is $\left\{x \in \mathbb{Q}^+ \middle| x \geq \frac{12}{5}\right\}$ where $\mathbb{Q}^+$ is the set of positive rational numbers.

## Exercise 3.5

1 Solve the following linear inequalities in the given domain.

a $5x+6 \leq 3x+20,\ x \in \mathbb{N}$

b $\frac{5}{3}x < -8(x-6),\ x \in \mathbb{Z}^+$

c $-2(12-2x) < 3x-24,\ x \in \mathbb{Q}^+$

d $\frac{3}{4}y+\frac{1}{6} > \frac{17}{10},\ x \in \mathbb{Z}$

e $4-\frac{5}{6}x > \frac{3}{2}x-8,\ x \in \mathbb{Q}$

f $-3(2x-3) > -(x-4)-4x,\ x \in \mathbb{Q}$

2 If five times a whole number increased by 3 is less than 33, then find all possible values of the number.

3 If the perimeter of a square field is less than 160 m, then find the possible dimensions of the field.

# 3.3 Graph of Linear Equations and Linear Inequalities

## 3.3.1 Graph of Linear Equations

In the previous grades, you have learnt how to draw the graphs of linear equations in two variables of the form $y = mx$ and in this grade you will learn how to draw the graph of any linear equation in two variables of the form $y = mx + b$.

### Activity 3.6

Draw the graphs of each of the following linear equations.

a $x = 3$

b $y = -4$

From your responses in Activity 3.6, observe that the graphs of equations of the form $x = a$ are vertical lines passing through the point $(a, 0)$ and the graphs of equations of the form $y = b$ are horizontal lines passing through the point $(0, b)$.

The graphs of linear equations in two variables are straight lines. Straight lines can be drawn by locating two points on the coordinate plane that satisfy the equation and drawing a line through these points.

### Definition 3.8

A linear equation in two variables $x$ and $y$ is an equation that can be reduced to the form $y = mx + b$, where $m$ and $b$ are numbers.

Given a linear equation in two variables $x$ and $y$: $y = mx + b$, where $m \neq 0$:

i $m$ is called the slope of the line representing the given equation;

ii $b$ is the y intercept of the line representing the given equation, that is, the line intersects y-axis at $(0, b)$ and

iii $\frac{-b}{m}$ is the x intercept of the line representing the given equation, that is, the line intersects x-axis at $\left(\frac{-b}{m}, 0\right)$.

### Example 3.19

Identify the slope, $y$ - intercept and $x$ - intercept of the lines represented by each of the following linear equations.

a $y - 3x = 4$

b $2y = 3x - 10$

c $3y = -4$

d $2x = 5$

**Solution**

a To find the slope, y-intercept and x-intercept of the line with equation $y - 3x = 4$, first rewrite the given equation as $y = 3x + 4$.

Then the slope of the line is 3, the $y$ - intercept is 4 and $x$ – intercept is $\frac{-3}{4}$

The solution for b, c and d are given the following table

| | Equation | $y = mx + b$ form | slope | y-intercept | x-intercept |
|---|---|---|---|---|---|
| b | $2y = 3x - 10$ | $y = \frac{3}{2}x + (-5)$ | $\frac{3}{2}$ | $-5$ | $\frac{10}{3}$ |
| c | $3y = -4$ | $y = \frac{-4}{3}$ | 0 | $\frac{-4}{3}$ | No |
| d | $2x = 5$ | – | undefined | No | $\frac{5}{2}$ |

## Note

- Given a linear equation $y = mx + b$, all points in the coordinate plane satisfying the given equation makes the line of the equation.

  You can draw the line representing the given equation by plotting two points of the line on the coordinate plane and drawing a line through the two points.

### Example 3.20

Draw the graph of each of the following linear equations.

a $y = x + 2$  b $y = 2x + 1$  c $2x + 3y = 2$

**Solution**

First, construct table of values by choosing some values, say 0 and 1 determine the corresponding values.

Next, plot the ordered pairs in the coordinate plane and then draw a line through these points.

a y=x+2

| $x$ | 0 | 1 |
|---|---|---|
| $y$ | 2 | 3 |
| $(x, y)$ | (0,2) | (1,3) |

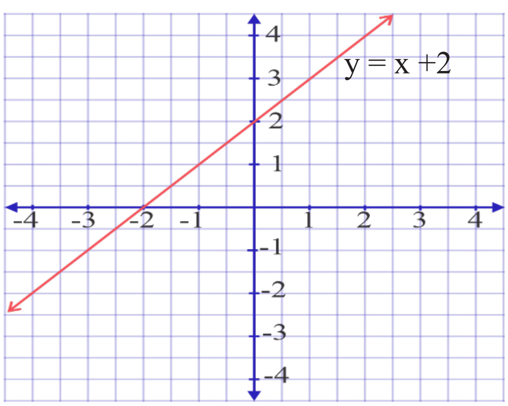



Figure 3.3

b $y=2x+1$

| x | 0 | 1 |
|---|---|---|
| y | 1 | 3 |
| (x, y) | (0,1) | (1,3) |

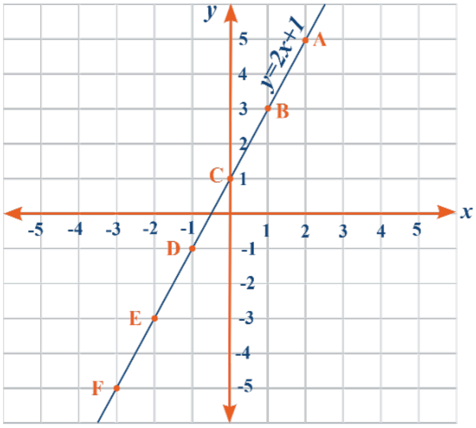


Figure 3.4

c $2x+3y=2$

| x | 0 | 1 |
|---|---|---|
| y | $\frac{2}{3}$ | 0 |
| (x, y) | $(0, \frac{2}{3})$ | (1, 0) |

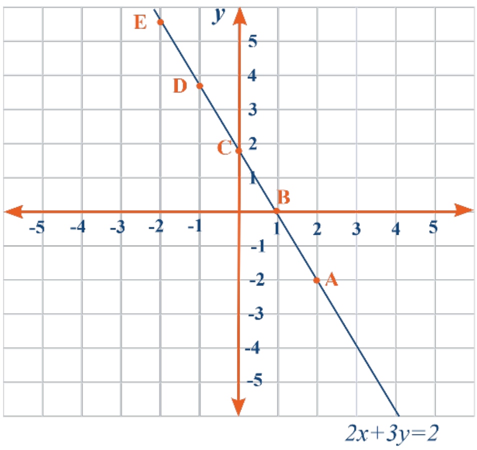


Figure 3.5

## Exercise 3.6

1 Draw the graph of each of the following linear equations on the coordinate plane.

a $y=3x-4$

b $y=-5x-2$

c $-2y=\frac{1}{2}x+3$

d $2y-0.25x=2$

e $-3y-6y-12$

f $\frac{-3}{2}y-2x=12$

2 Let the ratio of two numbers $\frac{x+1}{2}$ and y be 1:3. Then draw the graph of the equation that shows the ratio of these two numbers.

### 3.3.2 Graph of Linear Inequalities

In the previous discussion, you have learnt how to draw the graph of linear equations of the form $y = mx + b$. The graph of linear equations is a straight line.

In this part, you will learn how to draw the graphs of linear inequalities.

## Activity 3.7

a Draw the line $y = x + 2$

b Find at least five points that satisfy the linear inequality $y \leq x + 2$

c Locate these points obtained in (a) on the coordinate plane.

d Find the coordinates of at least five points that do not satisfy the inequality

e Locate the points obtained in (d) above on the coordinate plane.

f What do observe from c and e?

From your response in Activity 3.7, observe that the line $y = x + 2$ divided the coordinate plan into two parts and the coordinates of the points on one side of the line are all solution of the given inequality and the coordinates of the points on the other side of the line are not solutions.

Steps to draw the graph of the linear inequality $y \geq mx + b$.

**Step 1:** Draw the graph of the equation $y = mx + b$, which will be a straight line

**Step 2:** The line divides the plane into two different regions. Choose a point on one side of the line $y = mx + b$.

**Step 3:** If the coordinates of the chosen point satisfy the inequality $y \geq mx + b$, then the region containg the point is the graph of the given inequality.

If the coordinates of the chosen point do not satisfy the inequality $y \geq mx + b$, the region that does not contain the point is the graph of the given inequality

**Step 4:** Shade the region which contain the solutions of the inequality $y \geq mx + b$.

The line $y = mx + b$ is part of the shaded region and we draw the line with bold line as the inequality includes the equal sign, otherwise we use broken line to indicate the line is not part of the solution.

Similar steps can used to draw the graph of linear inequalities of the form: $y \leq mx + b$, $y > mx + b$ and $y < mx + b$.

## Note

For linear inequalities y $> mx + b$ and $y < mx + b$, the line $y = mx + b$ is not part of the graph; to show this, the graph of the line is a broken line.

## Example 3.21

Draw the graph of each of the following inequalities.

a $y \geq 2$

b $x < -2$

c $y \leq 2x + 1$

### Solution

a Consider the inequality $y \geq 2$

**Step 1:** First draw the line $y = 2$.

The line is a horizontal line through (0, 2) as in Figure 3.6.

The line $y = 2$ divides the region into two different regions

**Step 2:** Choose a point on one of the opposite side of the line $y = 2$.

Take a point (0.3), which is above the line and $3 \geq 2$ is true.

Then the region containing (0,3) is the graph of the given linear inequality

**Step 3:** Shade the parts of plane that contains (0,3).

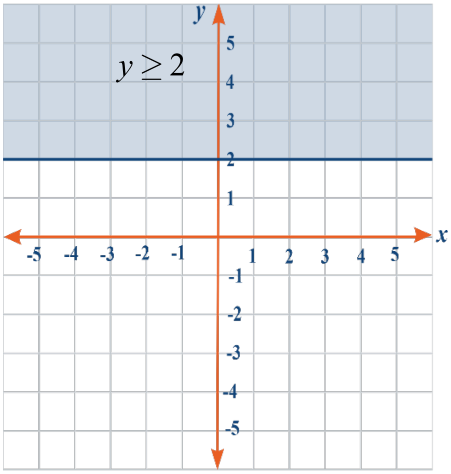



Figure 3.6

b Consider the inequality $x < -2$

**Step 1:** Draw the line $x = -2$

The line $x = -2$ is a vertical line through the (–2, 0) and it divides the plane in two different region as in Figure 3.7

**Step 2**: Choose a point on one of the opposite side of the line $x = -2$

Let us take the point (–3, 0) to the left of the line which satisfies $x < -2$

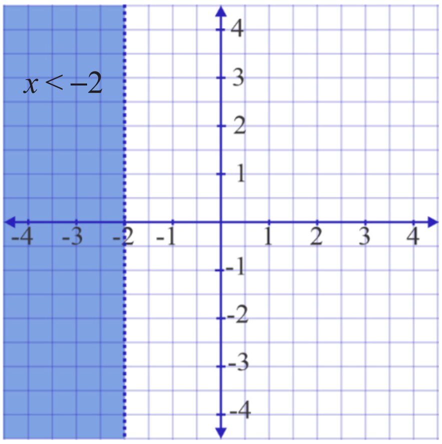



Figure 3.7

**Step 3:** Shade the half plane on the side of the point (– 3, 0).

The line $x = -2$ is not part of the solution and it is drawn using a broken line.

c $y \leq 2x + 1$

**Step 1:** Draw line $y = 2x + 1$ using the points (0,1) and (1,3) as in Figure 3.9.

The line $y = 2x + 1$ is part of the solution and draw using solid line.

**Step 2:** Choose a point on either of the opposite sides of the line $y = 2x + 1$.

Let us take a point $(1, \ -2)$ to the left side of the line $y = 2x + 1$.

Thus, the region containing the point (–2, 0) is not in the region that represents the given inequality

Therefore, the region that does not contain (-2, 0) is the graph of the given inequality.

**Step 3:** Shade the half plane on the other side of the point (-2, 0).

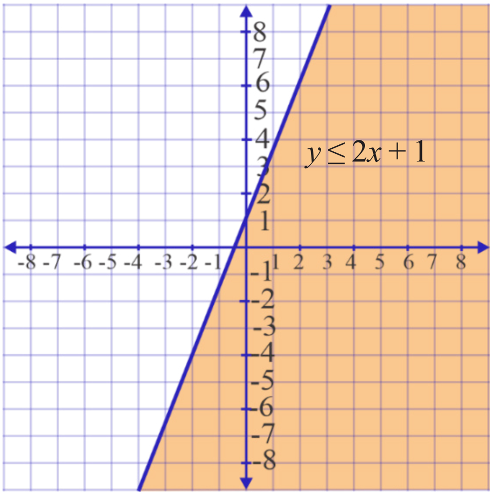


Figure 3.8

## Exercise 3.7

Draw the graph of each of the following linear inequalities.

a $y \leq x + 3$

b $y \geq -2x + 2$

c $2y > 4x + 3$

d $3y - 6x \leq 9$

e $\frac{1}{3}y < -x + 2$

f $\frac{1}{6}y > \frac{1}{3}x + \frac{1}{2}$

# Unit Summary

1 An algebraic expression which involves any of the inequality signs $<$, $>$, $\leq$, $\geq$ or $\neq$ is called inequality.

2 Linear equations (or inequalities) may involve one or more variables.

| Linear equations in one variable | Linear inequalities in one variable |
|---|---|
| $ax + b = 0$ | ♦ $ax + b > 0$, $ax + b < 0$<br>♦ $ax + b \geq 0$, $ax + b \leq 0$ |
| Linear equations in two variables | Linear inequalities in two variables |
| $ax + by + c = 0$ | ♦ $ax + by + c > 0$, $ax + by + c < 0$<br>♦ $ax + by + c \geq 0$, $ax + by + c \leq 0$ |

3 Linear equation in one variable can have at most one solution, and linear inequality in one variable may have one solution or no solution or infinitely many solutions in the set of rational numbers.

4 Two linear inequalities with the same solution set are solid to be equivalent inequalities.

5 Multiplying or dividing an inequality by negative numbers change the inequality sign.

## Review Exercises

1 Solve each of the following linear equations.

a $4(2x - 10) = 70 + 6x$

b $\frac{2x + 7}{3} - \frac{x - 9}{2} = \frac{5}{2}$

c $2(6y - 18) - 102 = 78 - 18(y + 2)$

d $12 - \frac{x - 7}{2} = \frac{x - 9}{4} + \frac{5x - 4}{2}$

e $0.78 - \frac{1}{25}x = \frac{3}{5}x - 0.5$

2 Solve each of the following inequalities.

a $-14y - 6 \leq 6y$

b $4x + 2 < 6x + 8$

c $8 - 6(x - 3) > -4x + 12$

d $\frac{7}{6}x + \frac{4}{3} \geq \frac{11}{6}x - \frac{7}{6}$

e $6 - 8(y + 3) > -(2y - 5) + 13$

f $-2 - \frac{w}{4} \leq \frac{1 + w}{3}$

g $-0.703 < 0.122x - 2.472$

h $3.88 - 1.335t \geq 5.66$

3 Find the solution set of the following inequalities in the given domain.

a $4x - \frac{1}{3} < 6x + 4\frac{2}{3},\ x \in \mathbb{W}$

b $9x - 4 < 13x - 7,\ x \in \mathbb{Z}$

c $0.7(x+3) < 0.4\ (x+3),\ x \in \mathbb{Q}^{+}$

d $3(x+2) - (2x-7) \leq (5x-1) - 2(x+6),\ x \in \mathbb{N}$

e $6 - 8\ (y+3) + 5y > 5y - (2y-5) + 13,\ x \in \mathbb{Q}$

4 Draw the graph of each of the following linear equations and linear inequalities

a $y = x - 3$

b $-3y - 9x = 12$

c $6y - 4x = 9$

d $y \geq 4x - 6$

e $4y - 8x < 12$

f $\frac{y}{2} > x - 3$

5 Find all the possible values of $x$ for which the perimeter of a rectangular field with length $x$and width $x - 60$ is at most 320 meters

6 A farmer planned to produce over 26,000 kg of maize, wheat and teff by the end of the year. He already harvested 10,200 kg of maize and 7,100 kg of teff, but the harvesting process of wheat is ongoing. How many kilograms of wheat should he expect to achieve his yearly plan?

7 If the minimum driving speed limit and maximum driving limit of certain road are 60 kilometers per hour and 70 kilometers per hour, what is the total time a car can take to drive 420 kilometers?

# Unit 4

# SIMILARITY OF FIGURES

## Learning outcomes:

**After completing this unit, you will be able to:**

- understand the concept of similar figures and related terminologies;
- identify the condition for which triangles are similar;
- apply similarity theorems to check whether two given triangles are similar or not;
- apply the concept of similarity of triangles in solving real life problems.

## *Key terms*

- plane figures
- constant of proportionality
- corresponding angles
- corresponding sides
- ratio of perimeter of similar figures
- ratio of area of similar figures
- similar figures
- SAS-similarity theorem
- AA-similarity theorem
- SSS-similarity theorem

## Introduction

If two plane figures have the same shape, we say that they are similar. The idea of similarity focuses only on shape of the geometric figures and it is being used in our day to day activities. In this unit, you will learn about similarity of plane figures and more emphasis will be given on similarity of triangles.

## 4.1 Similar Plane Figures

### Activity 4.1

1 Using the description below, identify which part have the same shape? Explain.

  a A 3 cm × 4cm photo and a 30 cm × 40 cm size of the same photo.

  b Two circles with different radius.

  c Two squares with different side lengths.

  d An object and its reflection through mirrors.

  e A square and equilateral triangle with equal side length..

2 Give an example of pairs of plane figures which have:

  a The same shape and the same size

  b The same shape but different size

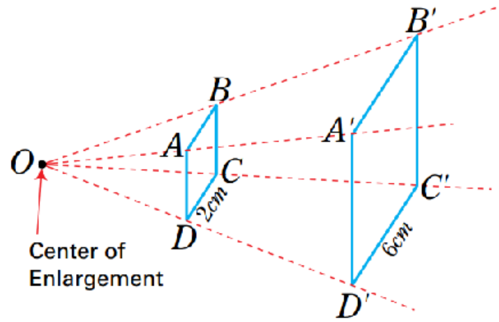
If two objects are similar, then they have exactly the same shape, but one of them is an enlargement of the other or smaller than the other without affecting its shape. Scale drawing shows an object exactly as it looks, but it is generally larger or smaller than the real object. Scale drawing show us better understanding about similar figures. In geometry, most of the time, we study similarity of plane figures like triangles, quadrilaterals, polygons, circles and other plane figures.

### Remark 4.1

1 The concept of similar figures does not consider color resemblance of the figures. It depends only in shape and size relationship of the objects.

2 Similar geometric figures are figures which have exactly the same shape, but not necessarily the same size.

In the figure below, square $A'B'C'D'$ is the image of square $ABCD$ of side 2cm under enlargement of scale factor 3, Then




$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'D'}{CD} = \frac{D'A'}{DA} = 3$$

$A'B' = B'C' = C'D' = D'A' = 6cm$

This implies that $A'B'C'D'$ is a square of side 6cm.

## Note

Scale factor is the number or the conversion factor which is used to change the size of a figure without changing its shape. It is used to increase or decrease the size of an object. Scale factor can be calculated if the dimensions of the original figure and the dimension of the changed (increased or decreased) figures are known. It is given as:

$$\text{Scale factor} = \frac{\text{length of new figure}}{\text{length of original figure}}$$

## Exercise 4.1

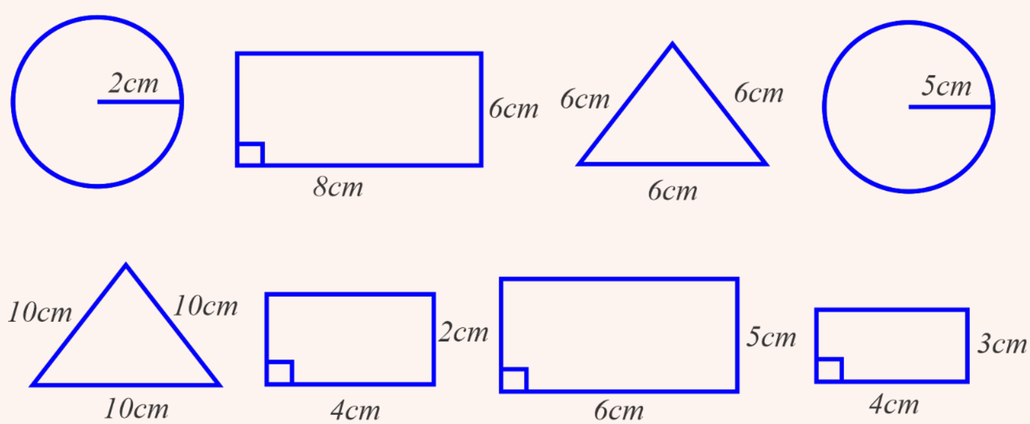
1 Suppose $A'$ and $A''$ are plane figures obtained by enlarging and diminishing another plane figure A by scale factor 2 and $\frac{1}{2}$ respectively,

a Are $A'$, $A''$ and A the same in shape? Why?

b Are $A'$, $A''$ and A equal in size? Why?

c What can you say about ratio of sides of these figures?

2 Which of the following geometric figures are similar in shape?



3 What is the length of the image of a 20cm long segment after enlargement with a scale factor 2?

4 Which members of the following families of plane figure are similar in shape? Why?

a Squares
b Rectangles
c Parallelograms
d Circles
e Equilateral triangles
f Isosceles triangles
g Trapeziums
h Rhombus

### 4.1.1 Definition and Illustration of Similar Figures

## Activity 4.2

Give at least two examples of plane figures whose:

a corresponding angles are congruent;
b corresponding sides are proportional;
c corresponding angles are congruent and corresponding sides are proportional.

From your responses in Activity 4.2, observe that two plane figures with corresponding angles congruent and corresponding sides are proportional are similar.

### Definition 4.1

Two polygons are said to be similar if the following three conditions are satisfied:

a the two polygons have the same number of sides;
b the corresponding angles of the two polygons are congruent and
c the corresponding pairs of sides of the two polygons are proportional, that is, the ratios of the lengths of all pairs of the corresponding sides are equal

That is, the polygons $A_1\ A_2 \cdots A_n$ and $B_1\ B_2 \cdots B_n$, with the same number of side are similar, written as: $A_1\ A_2 \cdots A_n \sim B_1\ B_2 \cdots B_n$ if

i $\angle A_1\ A_2\ A_3 \cong \angle B_1\ B_2\ B_3$, $\angle A_2\ A_3\ A_4 \cong \angle B_2\ B_3\ B_4$, …, $\angle A_nA_1A_2 \cong \angle B_nB_1B_2$ and

ii $\dfrac{A_1\ A_2}{B_1\ B_2} = \dfrac{A_2\ A_3}{B_2\ B_3} = \dfrac{A_3\ A_4}{B_3\ B_4} = \ldots = \dfrac{A_n\ A_1}{B_n\ B_1}$

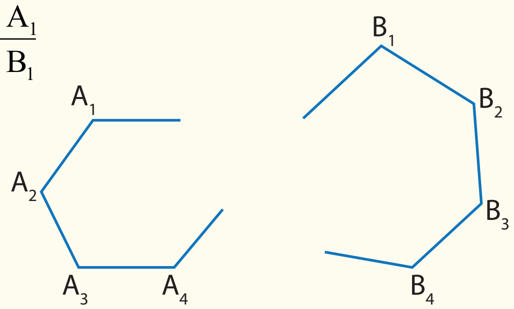



Figure 4.1

## Example 4.1

Show that any two squares are similar.

### Solution

Let ABCD and EFGH be two squares.

Then all the four angles of both squares are right angles and all the four sides of each of the two squares have equal lengths. That is,

i $\angle ABC \cong \angle EFG$, $\angle BCD \cong \angle FGH$, $\angle CDA \cong GHE$ and $\angle DAB \cong \angle HEF$ (the corresponding angles of the two squares are congruent);

ii $AB = BC = CD = DA = a$ and $EF = FG = GH = HE = b$; $\frac{AB}{EF} = \frac{BC}{FG} = \frac{CD}{GH} = \frac{DA}{HE} = \frac{a}{b}$ (the corresponding pairs of sides of the two squares are proportional).

This implies, ABCD and EFGH are similar, that is, $ABCD \sim EFGH$.

Thus, any two squares are similar.

## Example 4.2

Show that any two equilateral triangles are similar.

### Solution

Let ΔABC and ΔDEF be two equilateral triangles.

Each of all the three angles of each of the two triangles measure $60^0$ and all the three sides of each of the two triangles have equal lengths. That is,

i $\angle ABC \cong \angle DEF$, $\angle BCA \cong \angle FED$ and $\angle CAB \cong \angle FDE$ (the corresponding angles of the two squares are congruent);

ii $AB = BC = CA = a$ and $DE = EF = FD = b$; $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} = \frac{a}{b}$ (the corresponding pairs of sides of the two triangles are proportional).

This implies, ΔABC and ΔDEF are similar, that is, $ABC \sim DEF$.

Thus, any two equilateral trinagles are similar.

## Example 4.3

Let $\Delta ABC$ and $\Delta DEF$ be two triangles such that AB= 8cm, BC=9cm and AC= 10cm and DE=4cm, EF=5cm and DF=6cm. Then determine whether the pairs of the corresponding sides of $\Delta ABC$ and $\Delta DEF$ are proportional or not.

### Solution

From the given information, $\frac{AB}{DE} = \frac{8}{4} = 2$, $\frac{AC}{DF} = \frac{10}{6} = \frac{5}{3}$ and $\frac{BC}{EF} = \frac{9}{5}$

This implies the corresponding pairs of sides of $\Delta ABC$ and $\Delta DEF$ are not proportional.

## Example 4.4

Suppose the hexagons $ABCDEF$ and $A'B'C'D'E'F'$ shown in Figure 4.2 are similar. If $BC = 2cm$, $DE = 4cm$ and $B'C' = 3cm$, then find the length of $\overline{D'E'}$.

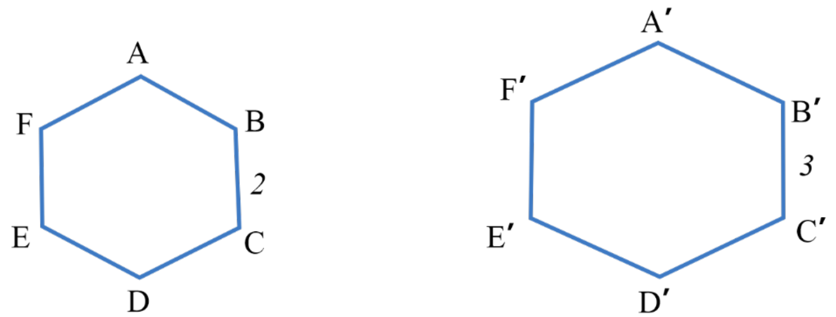



Figure 4.2 Hexagons

### Solution

It is given that, $ABCDEF \sim A'B'C'D'E'F'$.

Then the corresponding pairs of sides of the two hexagons are proportional.

That is, $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{2}{3} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EF}{E'F'} = \frac{FA}{F'A'}$

This implies, $\frac{DE}{D'E'} = \frac{2}{3}$, and then $D'E' = \frac{3}{2}DE = \frac{3 \times 4cm}{2} = 6cm$

## Exercise 4.2

1 Which of the following plane figures are always similar?

a Any two circles.

b Any two rectangles.

c Any two quadrilaterals.

2. The sides of a quadrilateral are 2cm, 4cm, 5cm and 6cm long. Find the lengths of all the sides of a similar quadrilateral whose longest side is 18cm long.

3. If $\Delta ABC \sim \Delta A'B'C'$ and AC = 20cm, A'C' = 15cm, B'C' = 12cm and A'B' = 9cm, then find the lengths of the remaining sides of $\Delta ABC$.

4. The sides of a pentagon are 6cm, 9cm, 10cm, 8cm and 11cm long. The shortest side of a similar pentagon is 12cm. Then find the lengths of the remaining sides of the second pentagon.

5. The length and width of a rectangular football field measures 100m and 72m respectively. A school wants to prepare a new football field similar to the old one with length 30m. What is the width of the new football field?

### 4.1.2 Similar Triangles

In this section, you will learn about similar triangles and theorems on similarity of triangles.

## Note

Two triangles, $\triangle ABC$ and $\triangle DEF$ are similar if their corresponding angles are congruent and their corresponding pairs of sides are proportional.

That is, $\triangle ABC$ and $\triangle DEF$ are similar, written as $\triangle ABC \sim \triangle DEF$ if

i $\angle ABC \cong \angle DEF$, $\angle CAB \cong \angle FDE$, $\angle BCA \cong \angle EFD$ (Corresponding angles are congruent) and

ii $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF}$ (corresponding pairs of sides are proportional)

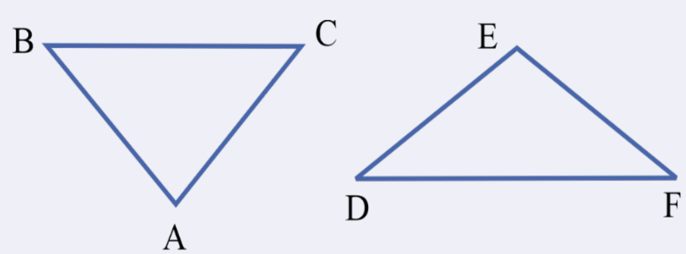



Figure 4.3

## Example 4.5

In Figure 4.4 , show that ΔABC and ΔLMN are similar.

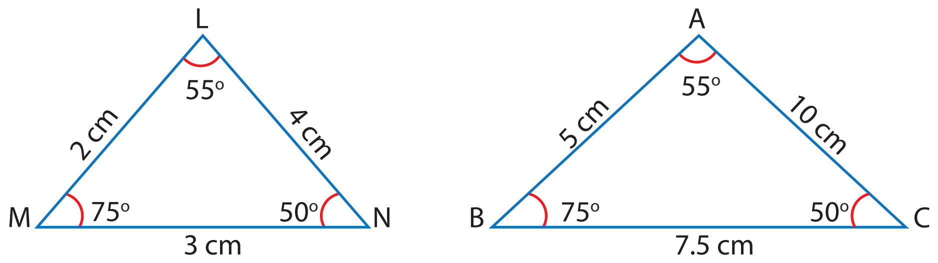



Figure 4.4

### Solution

i $m(\angle ABC) = m(\angle LMN) = 75^0$

$m(\angle BCA) \cong m(MNL) = 50^0$

$m(\angle CAB) = m(\angle NLM) = 55^0$

Therefore, the corresponding angles of △ABC and △LMN are congruent.

ii $\frac{AB}{LM} = \frac{5cm}{2cm} = 2.5$ , $\frac{BC}{MN} = \frac{7.5cm}{3cm} = 2.5$ and $\frac{AC}{LN} = \frac{10cm}{4cm} = 2.5$

Thus, the corresponding pairs of sides are proportional with constant of proportionality equal to 2.5. Therefore, ΔABC ~ ΔLMN

## Example 4.6

In the Figure 4.5, $\Delta ABE \sim \Delta ADC$, If $AB = 4cm$, $AE = 7cm$, $BE = 5cm$ and $DC = 20cm$, then find the lengths of ED and BC

### Solution

Given △ABE ~ △ADC.

By the definition of similarity of triangles, the corresponding sides are proportional.

That is, $\frac{AB}{AD} = \frac{BE}{DC} = \frac{AE}{AC}$

$\frac{4cm}{AD} = \frac{5cm}{20cm} = \frac{7cm}{AC}$

$\frac{4cm}{AE+ED} = \frac{1}{4} = \frac{7cm}{AB+BC}$

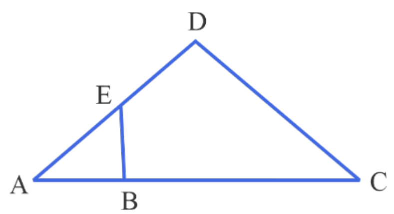



Figure 4.5

This implies

$$\frac{4cm}{7cm + ED} = \frac{1}{4} = \frac{7cm}{4cm + BC}$$

$$\frac{4cm}{7cm + ED} = \frac{1}{4} \text{ and } \frac{1}{4} = \frac{7cm}{4cm + BC}$$

Thus, we have

$7cm + ED = 16cm$ and $4cm + BC = 28cm$

Therefore, $ED = 9cm$ and $BC = 24cm$

## Note

Similarity of triangles is transitive relation. That is, If $\Delta ABC \sim \Delta DEF$ and $\Delta DEF \sim \Delta XYZ$, then $\Delta ABC \sim \Delta XYZ$.

## Exercise 4.3

1 In Figure 4.6 $\Delta ABC \sim \Delta DEF$. Find

a $m(\angle EFD)$ b $m(\angle DEF)$ c $m(\angle ABC)$ d $BC$

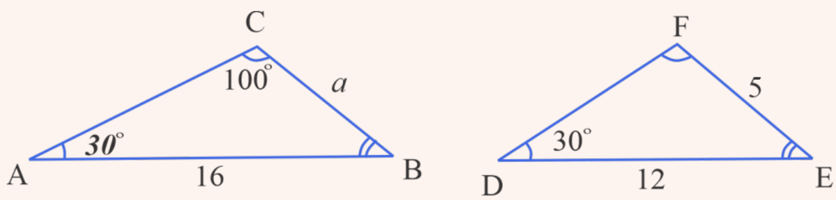



Figure 4.6

2 If $\Delta ABC \sim \Delta XYZ$ and $AC = 10cm$, $AB = 8cm$ and $XY = 4cm$, then find $XZ$.

3 In Figure 4.7 below, if $\Delta ABC \sim \Delta XBZ$ with $XB = 5cm$, $BZ = 5cm$, $CX = 8cm$ and $AC = 7cm$ ., then find the length of:

a $\overline{BA}$

b $\overline{XZ}$

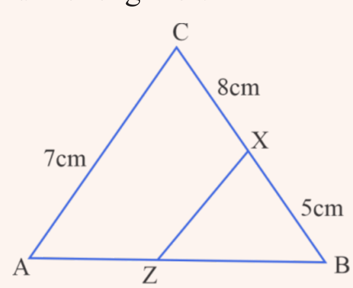



Figure 4.7

4 Suppose $\Delta DEF$ is obtained from $\Delta XYZ$ by the scale factor 6 units with $DE = 7cm$, $EF = 12cm$ and $XZ = 36\ cm$. Then, find the lengths of XY and YZ.

5 If $\Delta DEF \sim \Delta KLM$ such that $DE = (2x+2)$cm, $DF = (5x - 7)$cm, $KL = 2$cm, $KM = 3$cm and $EF = 10$cm, then find the length of $\overline{LM}$.

6 The sides of a triangle are 4cm, 6cm, and x cm long. The corresponding sides of a triangle similar to the first triangle are y cm, 12 cm and 8 cm respectively. What are the possible lengths of *x* and *y*? Are the values of *x* and *y* unique? Why?

7 In the Figure 4.8, if $\Delta XYZ \sim \Delta WYP$, express *d* in terms of *a*, *b* and *c*

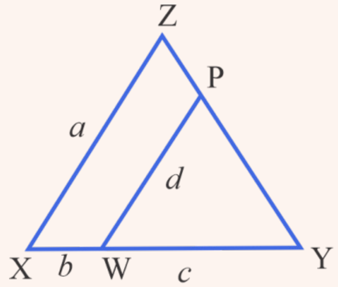



Figure 4.8

## 4.2 Tests for Similarity of Triangles

In this section, you will learn how to determine similarity of two triangles by using less conditions than the definition of similarity

### Activity 4.3

Using ruler and protractor, construct $\Delta ABC$ and $\Delta XYZ$ so that $m(\angle BAC) = m(\angle YXZ) = 30^0$ and $m(<ABC) = m(\angle XYZ) = 70^0$.

a What can you say about $\angle BCA$ and $\angle YZX$? Why?

b What are proportion of the pairs of the corresponding sides of the two triangles

From your responds in Activity 4.3 observe that the two triangles are similar with the given condition.

**Theorem 4.1 (Angle-Angle (AA) Similarity Theorem)**

If two angles of one triangle are congruent to the corresponding two angles of another triangle, then the triangles are similar. That is, for two triangles $\Delta ABC$ and $\Delta DEF$ if $\angle ABC \cong \angle DEF$ and $\angle BCA \cong \angle EFD$, then $\Delta ABC \sim \Delta DEF$

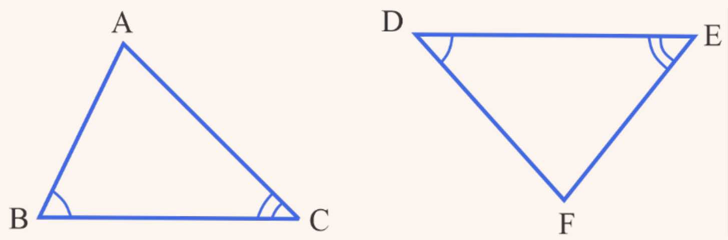



Figure 4.9

## Example 4.7

In Figure 4.10, if the sum of the measures of all the angles of ΔDEF is 180°, then show that ΔABC ~ ΔDEF

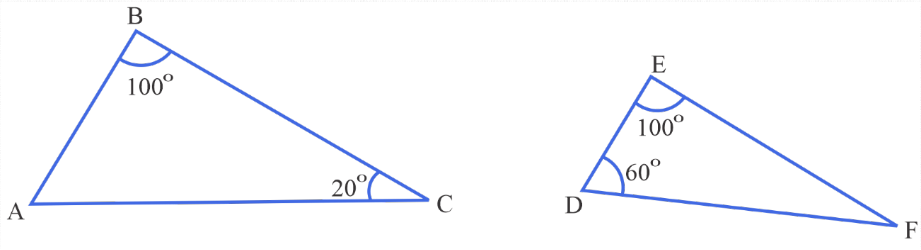



Figure 4.10

### Solution

In $\Delta$ ABC and $\Delta$ DEF,

a $\angle ABC \cong \angle DEF$ (Given)

b $m(\angle DFE) + 100° + 60° = 180°$ (Given)

c $m(\angle DFE) = 20°$ (From Step (b))

d $\angle ACB \cong \angle DFE$ (From Step (c))

Therefore, $\Delta$ ABC ~ $\Delta$ DEF (by AA-Similarity Theorem)

## Example 4.8

In Figure 4.11, if $\overline{A'C'}$ is parallel to $\overline{AC}$, then show that $\Delta$ ABC ~ $\Delta$ A'BC'

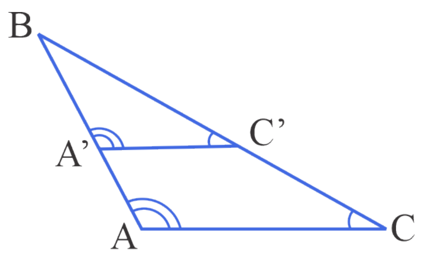



Figure 4.11

### Solution

i $\angle ABC \cong \angle A'BC'$ (Common angle of $\Delta$ ABC and $\Delta$ A'BC')

ii $\angle BAC \cong \angle BA'C'$ (Corresponding angles are congruent).

Therefore, $\Delta$ ABC ~ $\Delta$ A'BC' by AA similarity theorem

## Example 4.9

As shown in Figure 4.12, a long tree casts a shadow of length 10m and at the same time, a tree of height 2m casts a shadow of 5m. Then determine the height of the longer tree.

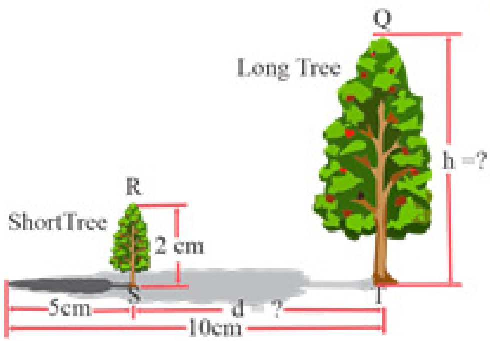
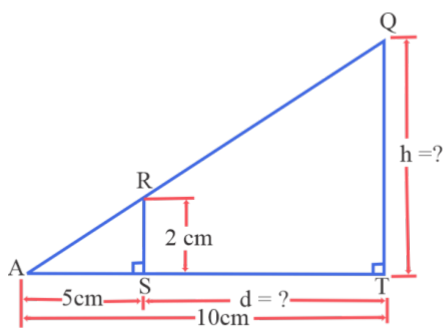
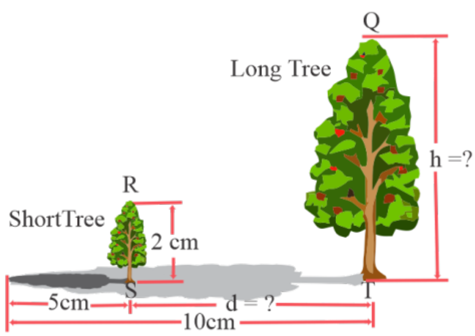



Figure 4.12

### Solution

Let $h$ be height of the longer tree. Then we have the following diagram.




a ∠QTA≅∠RSA as the two trees are perpendicular to the ground

b ∠QAT is common angle of the two triangles.

By AA-similarity Theorem, △ASR ~ △ATQ.

Then the pairs of the corresponding sides are proportional

$$\frac{AS}{AT} = \frac{SR}{TQ} = \frac{AR}{AQ}$$

$$\frac{5m}{10m} = \frac{1}{2} = \frac{2m}{h}$$

Therefore, height of the longer tree is $h = 4m$.

## Activity 4.4

Using ruler and protractor, construct $\Delta ABC$ and $\Delta XYZ$ so that $m(<CAB) = m(\angle ZXY) = 40^0$ and $AB = 2DE = 6cm$ and $AC = 2DF = 2cm$. By measuring the remaining angles and side length of the triangles, check whether the two triangles are similar or not.

From your response in Activity 4.4 observe that the two triangles are similar

### Theorem 4.2 (Side-Angle-Side (SAS) Similarity Theorem)

If two pairs of corresponding sides of two triangles are proportional and the included angles between these sides are congruent, then the two triangles are similar.

That is, given two triangles, $\triangle ABC$ and $\triangle DEF$, if If $\frac{AB}{DE} = \frac{BC}{EF}$ and $\angle ABC \cong \angle DEF$, then $\triangle ABC \sim \triangle DEF$

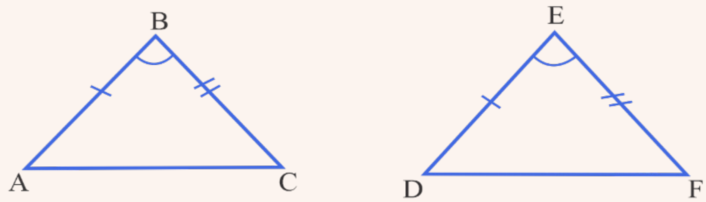



Figure 4.13

## Example 4.10

In Figure 4.14, let D and E be mid points of $\overline{AC}$ and $\overline{BC}$ respectively. Show that $\triangle ACB \sim \triangle DCE$.

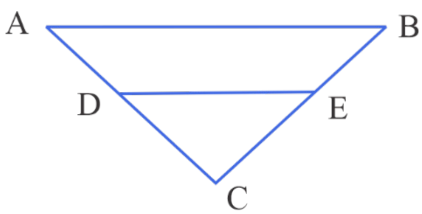



Figure 4.14

### Solution

Since D and E are mid points of $\overline{AC}$ and $\overline{BC}$ respectively, we have $AC = 2DC$ and $BC = 2EC$.

That is, $\frac{AC}{DC} = \frac{BC}{EC} = 2$ and $\angle ACB$ is common angle for the two triangles.

Therefore, $\triangle ACB \sim \triangle DCE$. by SAS similarity theorem

## Activity 4.5

Using ruler and protractor, construct ΔABC and ΔXYZ with the following measures:

AB = 6cm, AC = 8cm, BC = 10cm, XY = 3cm, XZ = 4cm and YZ = 5cm. Then, by measuring the angles of the triangles check whether the two triangles are similar or not.

From your response in Activity 4.5 observe that the two triangles are similar.

### Theorem 4.3 ( Side-Side-Side (SSS) Similarity Theorem)

If the three sides of one triangle are proportional to the corresponding three sides of another triangle, then the two triangles are similar.

That is, given ΔABC and ΔDEF, If $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$, then $\Delta ABC \sim \Delta DEF$

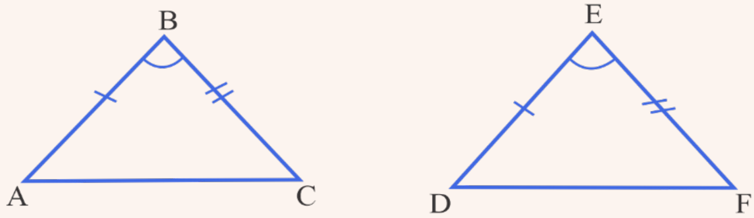



Figure 4.15

## Example 4.11

In Figure 4.16, if AB = 21 cm, $BC = 28cm$, $AC = 14cm$, $DE = 10cm$, $EF = 15cm$ and $DF = 20cm$, then show that $\Delta ABC \sim \Delta EFD$.

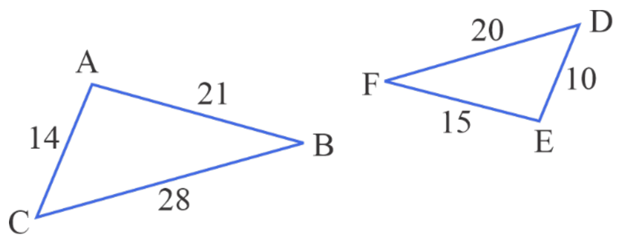



Figure 4.16

### Solution

From the given information, you have

$$\frac{BC}{FD} = \frac{28}{20} = \frac{7}{5}, \frac{BA}{FE} = \frac{21}{15} = \frac{7}{5} \text{ and } \frac{AC}{ED} = \frac{14}{10} = \frac{7}{5}$$

This implies that the pairs of the corresponding sides of △ABC and △EFD are proportional

△ABC ~ △EFD by the SSS similarity theorem.

## Example 4.12

In the Figure 4.17, if D, E and F are mid points of $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ respectively, then show that ΔABC ~ ΔEFD.

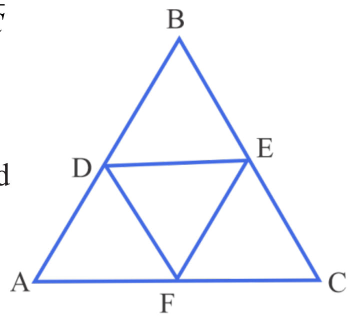


Figure 4.17

**Solution**

Suppose points D, E and F are mid points of $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ respectively.

Then $EF = \frac{1}{2}AB$, $DF = \frac{1}{2}BC$ and $DE = \frac{1}{2}AC$

This implies $\frac{AB}{EF} = \frac{BC}{DF} = \frac{AC}{DE} = 2$.

The ratios of the pairs of corresponding three sides of the two triangles are equal.

Therefore, ΔABC ~ ΔEFD by SSS similarity theorem.

## Exercise 4.4

1 In Figures 4.18, if $m(\angle WAS) = m(\angle BVS) = 45^0$, then show that ΔSAW ~ ΔSVB.

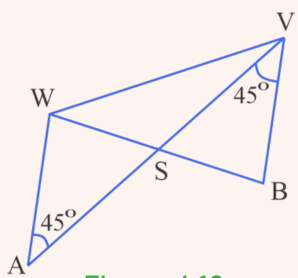


Figure 4.18

2 In Figure 4.19, ΔABC and ΔADE are similar. Find the values of $x$ and $y$.

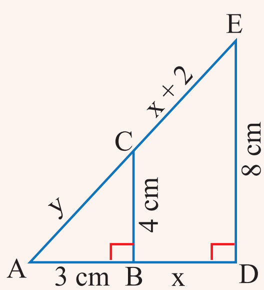


Figure 4.19

3 In Figure 4.20, if $\overline{AB} \| \overline{DE}$, AC = 20cm, AB = 16cm, BC = 24cm and D is a point on AC with CD = 15cm and E is a point on BC with CE = 18cm, then:

a Show that ΔDEC ~ ΔABC.

b How long is $\overline{DE}$?

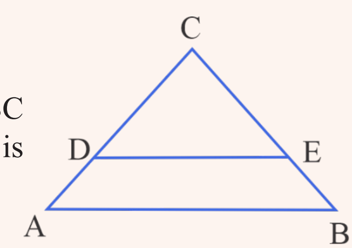
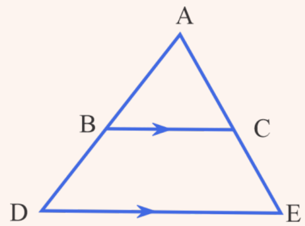


Figure 4.20

4 If the three sides of a triangle are 18cm, 14cm and 12cm long and each side of a second triangle is twice as long as the corresponding side of the first triangle, are the two triangles similar? Why?

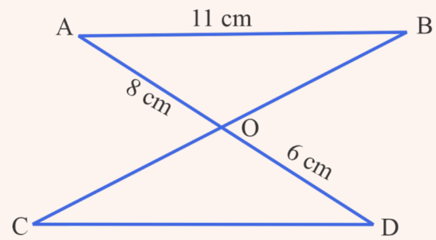
5 In the figure below, if $\overline{BC} \parallel \overline{DE}$, then show that $\triangle ABC \sim \triangle ADE$




6 For any triangle ABC, if $\angle BAC \equiv \angle ABC$, then show that $\Delta ABC \sim \Delta BAC$

7 Show that a diagonal of a rectangle divides the rectangle into two similar triangles.

8 In the figure below, if $\overline{AB}$ is parallel to $\overline{CD}$, $\overline{AD}$ and $\overline{BC}$ intersect at point O, $AB = 11$ cm, $AO = 8$ cm, $OD = 6$ cm, then find the length of CD.




9 On a leveled ground, the base of a tree is 20m far from the bottom of a 4.8m long flagpole. At a certain time, their shadows end at the same point, which is 60m far from the base of the flagpole. Find the height of the tree.

## 4.3 Perimeter and Area of similar Triangles

In the previous grade, you have learned how to find the perimeter and area of some special plane figures such as triangles, rectangles, squares, parallelograms and trapeziums. In this section, you will learn the proportionality of the area and perimeter of similar triangles with their corresponding sides.

### Activity 4.6

1 Let the lengths of the sides of $\Delta$ ABC be 3cm, 4cm and 5cm and the lengths of the sides of $\Delta$ DEF are 6cm, 8cm and 10cm. Then,

Compare the ratios of:

a their corresponding sides to the ratio of their perimeters;

b their corresponding sides to the ratio of their areas.

2 What do you conclude from these?

Form your response in Activity 4.6, observe that if two triangles are similar, the ratio of the perimeter of the two triangles is the same as the ratio of the sides and the ratio of their areas is the square of the ratio of the sides.

**Theorem 4.4**

If the ratio of the corresponding pair of sides of two similar triangles is k, then

a the ratio of their perimeters is given by

$$\frac{P_1}{P_2} = \frac{S_1}{S_2} = k,$$

where $P_1$ and $S_1$ are perimeter and length of a side of the first triangle and $P_2$ and $S_2$ are perimeter and length of a side of the second triangle;

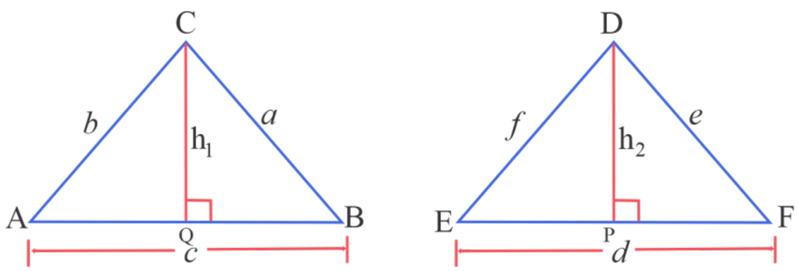
b the ratio of their areas is given by

$$\frac{A_1}{A_2} = \left(\frac{S_1}{S_2}\right)^2 = k^2$$

where, $A_1$ is area of the first triangle and $A_2$ is area of the second triangle.

**Proof**

Let ΔABC ~ ΔEFD with sides indicated in the figure below




a Given $\frac{AB}{EF} = \frac{BC}{FD} = \frac{AC}{ED} = k$

implies

$AB = k \times EF = kd$ that is $c = kd$

$BC = k \times FD = ke$ that is $a = ke$

$AC = k \times ED = kf$ that is $b = kf$

Then perimeter (P) is

$P(\Delta ABC) = AB + BC + AC = c + a + b = kd + ke + kf = k(d + e + f)$, and

$P(\Delta EFD) = EF + FD + ED = d + e + f$

Then,

$$\frac{P(\Delta ABC)}{P(\Delta EFD)} = \frac{c + a + b}{d + e + f} = \frac{k(d + e + f)}{d + e + f} = k$$

Hence, the ratio of the perimeters of the two similar triangles is $k$, which is equal to the ratio of the lengths of any pair of corresponding sides.

b Let $\overline{CQ}$ be the altitude of $\Delta ABC$ from vertex C to $\overline{AB}$ and $\overline{DP}$ be the altitude of $\Delta EFD$ from vertex D to $\overline{EF}$. Then

| | |
|---|---|
| $\angle CQB \cong \angle DPF$ | both are right angles |
| $\angle ABC \cong \angle EFD$ | corresponding angles of similar triangles |

Therefore,

| | |
|---|---|
| $\Delta CQB \sim \Delta DPF$ | by AA similarity theorem |
| $\frac{CQ}{DP} = \frac{CB}{DF}$ | by definition of similar triangles |

$$\frac{h_1}{h_2} = \frac{a}{e} = k$$

This implies $h_1 = kh_2$

The areas are given by

$$a(\Delta ABC) = \frac{1}{2}ch_1 = \frac{1}{2}(kd)\ (kh_2) = \frac{1}{2}(k^2)\ (dh_2), \text{ and}$$

$$a(\Delta EFD) = \frac{1}{2}dh_2$$

Hence,

$$\frac{a(\Delta ABC)}{a(\Delta EFD)} = \frac{\frac{1}{2}(k^2)\ (dh_2)}{\frac{1}{2}dh_2} = k^2$$

Therefore, the ratio of the areas of the two similar triangles is $k^2$, which is the square of the ratio of the lengths of any pair of corresponding sides.

## Example 4.13

Find the ratio of the areas of two similar triangles,

a if the ratio of the corresponding sides is $\frac{5}{4}$.

b if the ratio of their perimeters is $\frac{2}{5}$.

### Solution

Let $A_1$ and $A_2$ be the areas of the two similar triangles and $P_1$ and $P_2$ be the perimeters of the two triangles and $S_1$ and $S_2$ be lengths of their corresponding sides. Then a $\frac{A_1}{A_2} = \left(\frac{S_1}{S_2}\right)^2 = \left(\frac{5}{4}\right)^2 = \frac{25}{16}$

b $\frac{A_1}{A_2} = \left(\frac{S_1}{S_2}\right)^2 = \left(\frac{P_1}{P_2}\right)^2 = \left(\frac{2}{5}\right)^2 = \frac{4}{25}$

## Example 4.14

Let the areas of two similar triangles be 80 cm$^2$ and 5cm$^2$ respectively. If a side of a smaller triangle is 2cm long, then find

a the length of the corresponding side of the larger triangle.

b the ratio of the perimeters of the larger triangle to the smaller triangle.

### Solution

a Let $A_1$ be the area of the larger triangle and $A_2$ be area of the smaller triangle and $S_1$ and $S_2$ be the lengths of their corresponding sides respectively. Then

$$\frac{A_1}{A_2} = \left(\frac{S_1}{S_2}\right)^2$$

$$\frac{80}{5} = \left(\frac{S_1}{2\text{cm}}\right)^2$$

$$\frac{80}{5} = \frac{S_1^2}{4}$$

$$S_1^2 = \frac{320}{5} = 64$$

$$S_1 = \sqrt{64} = 8\text{cm}$$

Therefore, the corresponding side of the larger triangle is $S_1 = 8$cm

b Let $P_1$ and $P_2$ be perimeters of the larger and smaller triangles respectively. Then

$\frac{P_1}{P_2} = \frac{S_1}{S_2} = \frac{8cm}{2cm} = 4$, the ratio of the perimeters of the larger triangle to the the smaller triangle.

## Exercise 4.5

1 The sum of the perimeters of two similar triangles is 18cm and the ratio of their corresponding sides is 4:5. Find the perimeter of each triangle.

2 Let the areas of two similar triangles be $144cm^2$ and $81cm^2$ respectively. Then

a find the ratio of their perimeters.

b find the length of a side of the second triangle if the length of the corresponding side of the first triangle is 6 cm.

3 Two triangles are similar, where the length of a side of one is 2 units and the length of the corresponding side of the other triangle is 5 units.

a Find the ratio of their perimeters.

b Find the ratio of their areas.

4 For two similar triangles, find the ratio of:

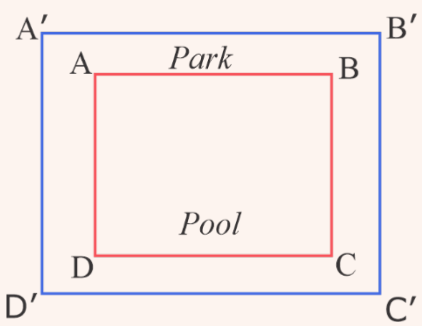
a their corresponding sides, if their areas are $38cm^2$ and $62cm^2$ respectively

b their perimeters, if their areas are $40cm^2$ and $25cm^2$ respectively

5 Two triangles are similar and the length of a side of one of the triangles is 6 times that of the length of corresponding side of the other. Find the ratio of their perimeters and the ratio of the areas of the triangles.

6 Given two triangles are similar and the length of a side of one of the triangles is 2 times that of the corresponding side of the other. If the area of the smaller triangle is 25sq.cm, then find the area of the larger triangle.

7 The figure below represents a swimming pool surrounded by a park. The two quadrilaterals are similar and the area of the pool is 100 $m^2$. What is the area of the park if the length of $\overline{A'B'}$ is four times the length of $\overline{AB}$?



# Unit Summary

1 Two or more objects are similar, if they have exactly the same shape, but not necessarily the same size. One is an enlargement or reduction of the other without affecting its shape.

2 If two angles of one triangle are congruent to the corresponding two angles of another triangle, then the two triangles are similar.

3 If two pairs of corresponding sides of the two triangles are proportional and the included angles between these sides are congruent, then the two triangles are similar.

4 If the three sides of one triangle are proportional to the corresponding three sides of another triangle, then the two triangles are similar.

5 The ratio of the perimeters of two similar triangles is the same as the ratio of the lengths their corresponding sides.

6 The ratio of areas of two similar triangles is the same as the square of the ratio of the corresponding perimeters of the triangles. The ratio of the areas of the two similar triangles is equal to the square of the ratio of their corresponding sides.

7 If the ratio of the corresponding sides of two similar polygons is 1, then the polygons are congruent.

## Review Exercises

1 State whether each of the following statements is true or false. Justify your answer.

a If two triangles are similar, then they are congruent

b If two triangles are congruent, then they are similar.

c All equilateral triangles are similar.

d All equilateral triangles are congruent.

e Any two isosceles triangles are similar.

f All rhombuses are similar.

2 Show that

a the diagonal of a square divides the square into two similar triangles.

b the altitude of an equilateral triangle divides the triangle into two similar right angled triangles.

3 Which of the following is equivalent to ΔADF~ΔLMN?

a ΔDFA ~ ΔNML

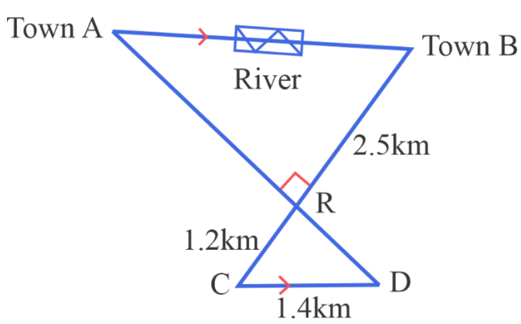
b ΔFAD ~ ΔMNL

c ΔAFD ~ ΔLMN

d ΔDAF ~ ΔMLN

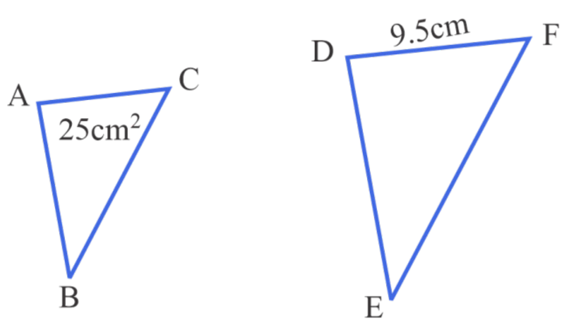
4 Suppose a rectangle ABCD is similar to rectangle PQRS, and $AB = 14$ cm, $BC = 8$ cm and $PQ = 21$ cm. Determine the length of QR.

5 A road construction company wants to connect two towns on opposite sides of a river with a road. Surveyors have laid out a map as shown below. The line joining town A to town B is parallel to the line joining C to D. What is the distance between the two towns?




6 Given that ΔABC ~ ΔXYZ, the area of ΔABC is 45 sq. units, the area of ΔXYZ is 80 sq. units and $YZ = 12$ units. Find BC

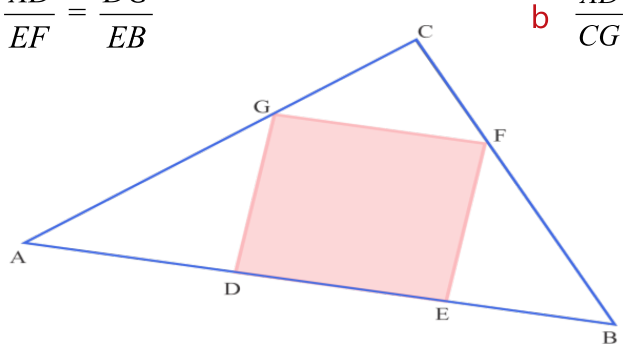
7 In the figure below, ΔABC ~ ΔDEF and the area of ΔABC is 25cm², $DF = 9.5$ cm and $AC = 7$ cm. Find the area of ΔDEF




8 In the figure below, DEFG is a square and ∠ACB is a right angle. Show that

a $\frac{AD}{EF} = \frac{DG}{EB}$

b $\frac{AD}{CG} = \frac{DG}{CF}$



# Unit 5

# THEOREMS ON TRIANGLES

## Learning outcomes:

**After completing this unit, you will be able to:**

- understand basic concepts about right angled triangles;
- apply theorems on right angled triangles;
- apply the concept of triangles in solving real-life problems.

### *Key terms*

- right angled triangle
- Pythagoras Theorem
- supplementary angles
- adjacent angles
- altitude
- median
- Angle Sum Theorem
- Euclid's Theorem
- complementary angles
- exterior angle
- midpoint

## Introduction

Triangles are applicable in many areas, for example in physics engineering etc.

In your previous grades, you have learnt different concepts about triangles. In this unit, you will learn about some basic theorems on triangles such as Angle Sum Theorem of a triangle, Euclid's Theorem and Pythagoras Theorem.

## 5.1 Theorems of Triangles

In this section, you will learn about concepts and theorems related with angles of triangles and sides of triangles. A triangle is a plane figure that has three sides and three angles. A triangle can be labeled with letters at the vertices. Thus, the triangle Figure 5.1 can be named as $\triangle BAC$ or $\triangle CBA$ or $\triangle ACB$.

The three angles of the triangle in Figure 5.1 are named as:

a $\angle BAC$, the angle with vertex at A and sides $\overline{AB}$ and $\overline{AC}$;

b $\angle CBA$, the angle with vertex at B and sides $\overline{BA}$ and $\overline{BC}$; and

c $\angle ACB$, the angle with vertex at C and sides $\overline{CA}$ and $\overline{CB}$.

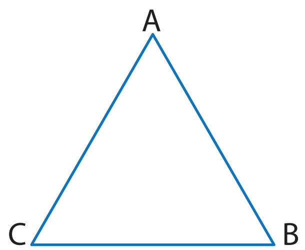



Figure 5.1 A triangle

### Note

1 The angle measure of a straight angle is $180^0$.

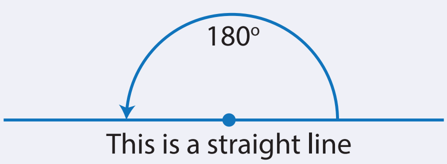



Figure 5.2

2 If the sum of the measures of two angles is $180^0$, then the two angles are called supplementary angles.

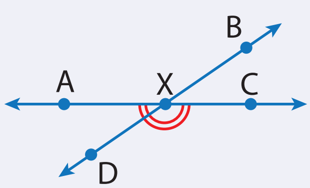



Figure 5.3

In Figure 5.3, given that $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ are straight lines, $m(\angle AXB) + m(\angle BXC) = 180^0$

Thus, $\angle AXB$ and $\angle BXC$ are supplementary angles.

3 If the sum of the measure of two angles is $90^0$, then the angles are called complementary angles.

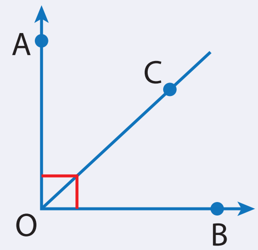



Figure 5.4

In Figure 5.4 given that ∠AOB is a right angle, m(∠AOC) + m(∠COB) = $90^0$

Therefore, ∠AOC and ∠COB are complementary angles.

4 If two angles have the same vertex and one common side with no common interior point, then the angles are called adjacent angles. In Figure 5.3 above ∠AXB and ∠AXD are adjacent angles, but ∠AXB and ∠AXC are not adjacent angles, because any point in ∠AXB is a common interior point for both angles.

## Example 5.1

In Figure 5.5 below, if ∠AOD is a straight angle, $x = \frac{1}{2}(y + z)$ and $z = 50^0$, then find the values of $x$ and $y$.

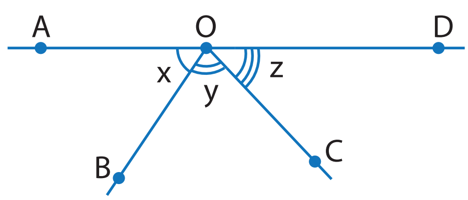



Figure 5.5

### Solution

It is given that $z = 50^o$ and $x = \frac{1}{2}(y + z)$

$x + y + z = 180^0$ (measure of a straight angle)

$\frac{1}{2}(y + 50^0) + y + 50^0 = 180^0$ (replacing $x$ by $\frac{1}{2}(y + 50^0)$ and $z = 50^0$)

$\frac{3y}{2} + 75^0 = 180^0$

$\frac{3y}{2} = 180^0 - 75^0 = 105^0$

$y = \frac{2}{3}(105^0)$

$y = 70^0$

Then $x = \frac{1}{2}(y + 50^0) = \frac{1}{2}(70^0 + 50^0) = 60^0$

## Example 5.2

If two angles are complementary and one of them is $\frac{3}{2}$ times the other, then what is the measure of each of the two angles?

**Solution**

Let θ and δ be the two complementary angles and $\theta = \frac{3}{2}\delta$.

Then, $\theta + \delta = 90^0$ (complementary angles)

$\frac{3}{2}\delta + \delta = 90^0$ (substitution)

$\frac{5}{2}\delta = 90^0$ (adding like terms)

$\delta = \frac{2}{5}(90^0) = 36^o$ (multiplying both sides by $\frac{2}{5}$)

From the relation $\theta + \delta = 90^0$, you have $\theta + 36^0 = 90^0$.

$\theta = 90^0 - 36^0 = 54^0$

Therefore, the measures of the two given complementary angles are $36^0$ and $54^0$.

### Basic Concepts Related to Triangles and Classifications of Triangles

Given a triangle:

1 Base of a triangle: any one of the three sides of a triangle can be taken as the base of the triangle.

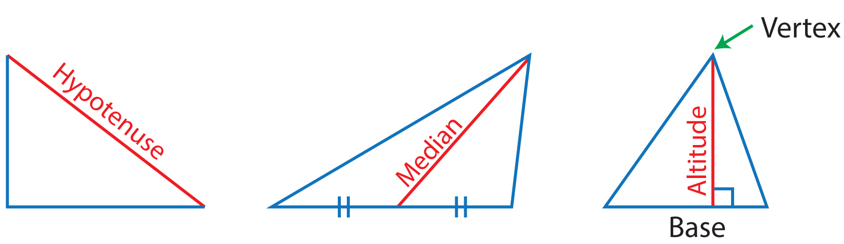
2 Vertex angle of a triangle: an angle opposite to the base of a triangle.

3 Altitude: a line segment drawn from any vertex perpendicular to the opposite side.

4 Median of a triangle: a line segment drawn from any vertex to the midpoint of the opposite side.

5 Angle bisector of a triangle: a line segment that divides (bisects) the vertex angle in to two congruent angles.

6 Hypotenuse of a right triangle: the longest side of the right triangle; it is the opposite side of the right angle.



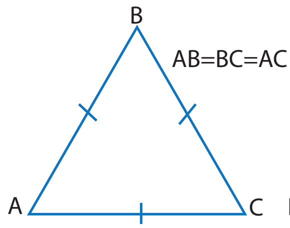
**Triangles are classified according to the lengths of their sides and measures of their interior angles**

I. According to their sides, triangles are classified as:

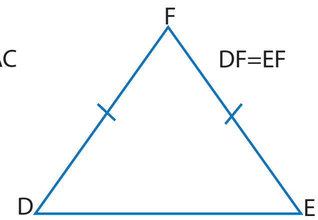
a **Equilateral triangles**: Triangles with three sides are congruent or equal in length

b **Isosceles triangles**: Triangles with two sides are congruent or equal in length

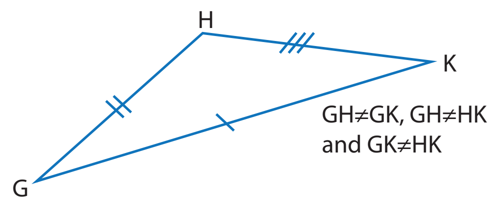
c **Scalene triangles**: Triangles with all three sides are different in length.



Equilateral Triangle



Isosceles Triangle



Scalene Triangle

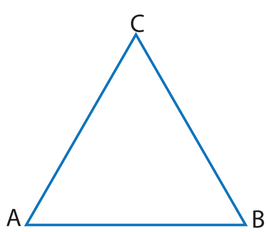
From the classification using sides, equilateral triangles are isosceles triangles.

II. According to their interior angles, triangles are classified as:

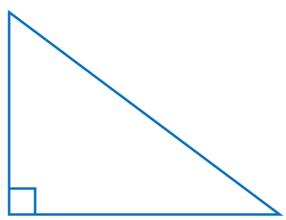
1 **Acute angled triangle**: all interior angles of a triangle measure less than $90^0$.

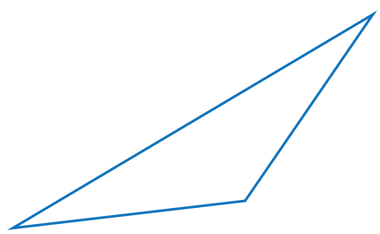
2 **Right angled triangle**: Triangles having one right angle.

3 **Obtuse angled triangle**: Triangles having one angle with measure greater than $90^0$.



Acute angled triangle


Right angled triangle


Obtuse angled triangle

**Theorem 5.1 (Isosceles Triangle Theorem)**

If two sides of a triangle are congruent, then the two angles opposite to the congruent sides are congruent. That is, in $\triangle ABC$, if $\overline{AB} \cong \overline{AC}$, then $\angle ABC \cong \angle ACB$.

**Proof:**

Consider $\Delta ABC$ in Figure 5.6. Suppose that $\overline{AB}$ and $\overline{AC}$ are congruent.

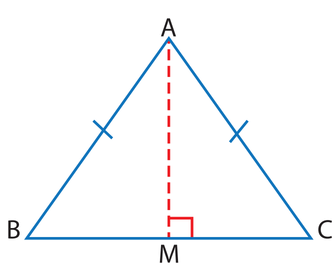


Figure 5.6

| | Statements | Reasons |
|---|---|---|
| i | $\overline{AB} \cong \overline{AC}$ | Given |
| ii | Draw an altitude $\overline{AM}$ from vertex A to side $\overline{BC}$ so that $m(\angle AMB) = m(\angle AMC) = 90^0$ | Construction |
| iii | $\overline{AM}$ is common side for $\Delta ABM$ and $\Delta ACM$ | |
| iv | $\Delta ABM \cong \Delta ACM$ | Right Angle - Hypotenuse-Side Theorem |
| v | $\angle ABC \cong \angle ACB$ | Property of Congruent Triangles |

## Example 5.3

In triangle $\Delta ABC$ in Figure 5.7 below, it is given that $AB = AC = 12cm$, $BC = 7cm$ and $m(\angle ABC) = 70^0$, then find the measure of $\angle ACB$.

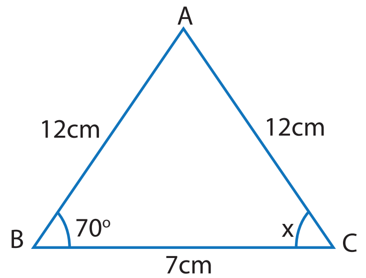



Figure 5.7

### Solution

In $\Delta ABC$, it is given that $AB = AC = 12cm$.

Thus, $\Delta ABC$ is isosceles triangle.

Then by the Isosceles Triangle Theorem, the base angles are congruent.

That is, $\angle ACB \cong \angle ABC$

Hence $m(\angle ACB) = m(\angle ABC) = 70^0$.

### Theorem 5.2 (Converse of Isosceles Triangle Theorem)

If two angles of a triangle are congruent, then the two sides opposite to the congruent angles are congruent. That is, in $\triangle ABC$, if $\angle ABC \cong \angle ACB$ then $\overline{AB} \cong \overline{AC}$.

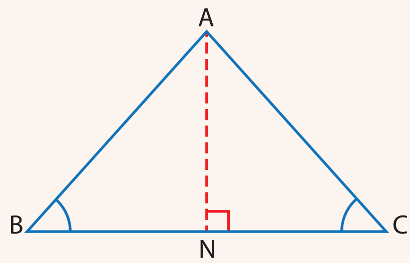



Figure 5.8

## Angle Sum Theorem of a Triangle

### Activity 5.1

Do the activity in a group of two or three students.

Using ruler and compass;

a draw a triangle as in Figure 5.9;

b cutout the two vertices of the triangle by a scissor carefully;

c connect the two vertex angles with the third angle side by side.

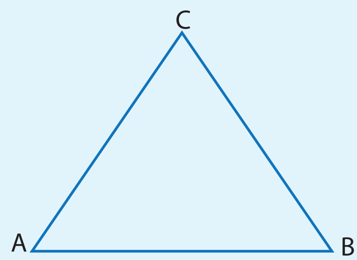



Figure 5.9

d What can you say about the sum of the measures of interior angles of the triangle?

From your responses in Activity 5.1, observe that the sum of the measures of interior angles of the triangle is $180^0$ and this result is true for any triangle, as stated in the following Theorem.

### Theorem 5.3 (Angle Sum Theorem)

The sum of the measures of all the three interior angles of a triangle is $180^0$

**Proof:**

Consider $\triangle ABC$ as in Figure 5.10.

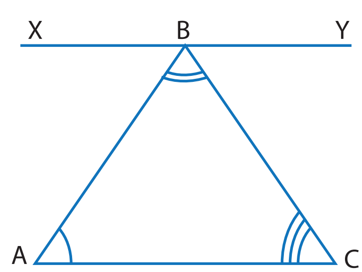



Figure 5.10

| | Statements | Reasons |
|---|---|---|
| 1 | Through vertex B, draw line $\overline{XY}$ parallel to $\overline{AC}$ | Through a point not on a line there is exactly one line parallel to the given line |
| 2 | $m(\angle ABC) + m(\angle CBY) + m(\angle ABX) = 180^0$ | measure of a straight angle |
| 3 | $\angle ABX \cong \angle CAB$ and $\angle BCA \cong \angle CBY$ | alternate interior angles are congruent |
| 4 | $m(\angle ABC) + m(\angle BCA) + m(\angle CAB) = 180^0$ | substitution |

Therefore, the sum of the measures of all the interior angles of a triangle is $180^0$.

## Example 5.4

In Figure 5.11, in $\triangle ABC$, $\angle BCD$ is a straight angle and $m(\angle ABC) = m(\angle BAC) = x$ and $m(\angle ACD) = 130^\circ$, find the value of $x$.

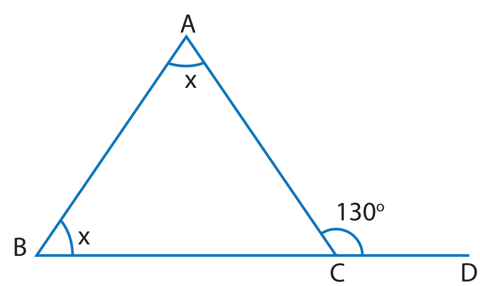


Figure 5.11

### Solution

| | Statements | Reasons |
|---|---|---|
| 1 | $m(\angle BCA) + m(\angle ACD) = 180^0$ | Supplementary angles |
| 2 | $m(\angle BCA) + 130^0 = 180^0$ | Given |
| 3 | $m(\angle BCA) = 180^0 - 130^0 = 50^0$ | |
| 4 | $m(\angle CBA) + m(\angle BAC) + m(\angle BCA) = 180^0$ | Angle Sum Theorem of a triangle |
| 5 | $x + x + 50^0 = 180^0$ | Given |
| 6 | $2x = 130^0$ | Adding like terms |
| 7 | $x = 65^0$ | Dividing both sides by 2. |

## Example 5.5

In an isosceles triangle, if the vertex angle measures $75^0$, find the measures of the two base angles.

**Solution**

Let one of the two congruent base angles of the triangle be θ. Then

$75^0 + \theta + \theta = 180^0$ (Angle Sum Theorem of a Triangle)

$75^0 + 2\theta = 180^0$ (Adding like terms)

$2\theta = 180^0 - 75^0 = 105^0$ (Subtracting $75^o$ from both sides)

$\theta = 52.5^0$ (Dividing both sides by 2)

Therefore, the measure of each base angle of the given triangle is $52.5^0$

## Example 5.6

Show that the measure of each angle of an equilateral triangle is $60^o$

**Solution**

Let $\Delta ABC$ be an equilateral triangle.

Then, $m(\angle ABC) = m(\angle ACB) = m(\angle BAC)$, by definition of an equilateral triangle.

By the Angle Sum Theorem of a triangle, $m(\angle ABC) + m(\angle ACB) + m(\angle BAC) = 180^o$

This implies $3m(\angle ABC) = 180^o$

$$m(\angle ABC) = \frac{180^o}{3} = 60^o$$

Thus, $m(\angle ABC) = m(\angle ACB) = m(\angle BAC) = 60^o$

## Exercise 5.1

1 Classify each of the following triangles as acute triangle, obtuse triangle or right triangle with the following measures of angles:

a $90°, 45°, 45°$
c $80°, 60°, 40°$
e $90°, 35°, 55°$
b $60°, 60°, 60°$
d $130°, 40°, 10°$
f $92°, 38°, 50°$

2 Classify each of the following triangles according to the lengths of the sides as equilateral, isosceles or scalene triangles with the given side lengths.

a 6 cm, 3 cm, 5 cm
d 8 cm, 12 cm, 10 cm
b 6 cm, 6 cm, 6 cm
e 3 cm, 4 cm, 5 cm
c 7 cm, 7 cm, 5 cm
f 3.5 cm, 3.5 cm, 4.5 cm

3 In $\triangle ABC$, if $\overline{AB} \cong \overline{BC}$ and $m(\angle ABC) = 120^0$, then find the measures of the angles $\angle CAB$ and $\angle ACB$.

4 Determine if it is possible to have a triangle with the following three angles as its interior (Give reasons in support of your answer).

a 110°, 60°, 30°

b 70°, 70°, 70°

c 80°, 35°, 65°

d 50°, 50°, 90°

5 Identify each of the following statements as true or false and give your reasons.

a The sum of interior angles of some scalene triangles can be greater than $180^{0.}$

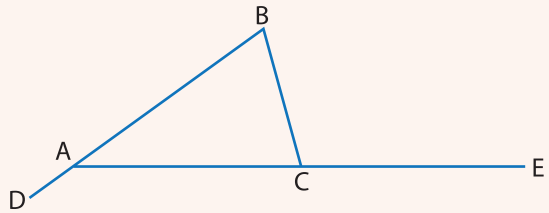
b An obtuse angled triangle can an equilateral triangle.

c An equilateral triangle can be a right angled triangle.

d Every equilateral triangle is an isosceles triangle.

6 Show that the angle bisector of an equilateral triangle is perpendicular to the base.

7 In the figure below, if $\overline{AB} \cong \overline{AC}$ and $m(\angle BCE) = 100^0$, then find $m(\angle DAC)$




8 In Figure 5.12, find the values of $x$ and $y$.

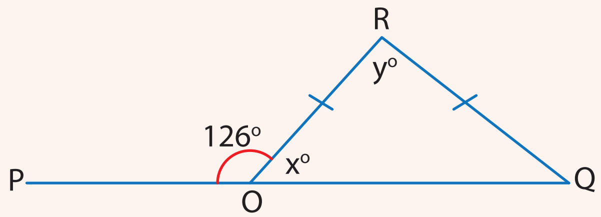



Figure 5.12

9 In Figure 5.13 below, $\Delta BAC$ is a right angled triangle with right angle at A and D is a point on $\overline{BC}$. If $m(\angle ACB) = 50^0$ and $m(\angle DAC) = 45^0$, then find $m(\angle ABD)$ and $m(\angle ADC)$.

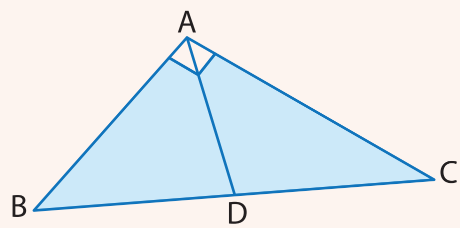
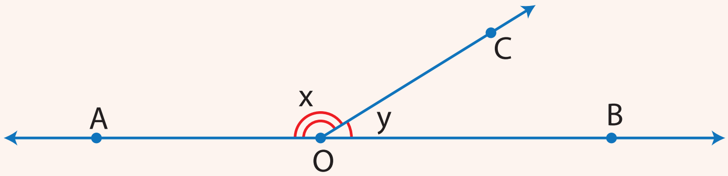



Figure 5.13

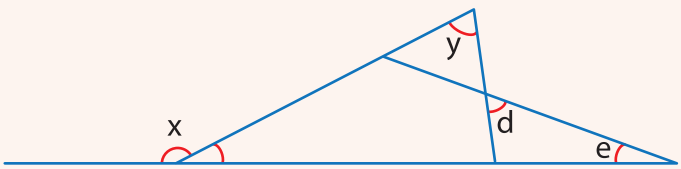
10 If x and y are supplementary angles and $x = \frac{2}{3}y$, then find the values of x and y.

11 If one interior angle of a triangle is twice of the second angle and three times of the third angle, then what is the measure of each angle of the triangle?

12 In the figure below, ∠AOB is a straight angle, then find the value of $x$ if $x - y = 35^0$.




13 In the figure below, express $y$ in terms of $d$, $e$ and $x$.

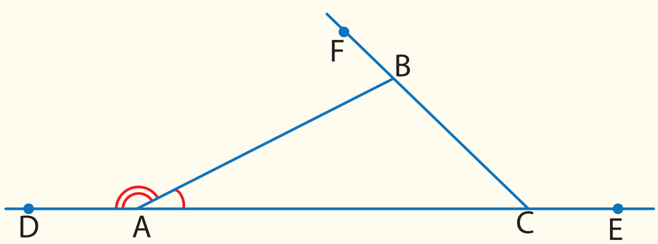



## 5.2 Exterior Angles of a Triangle

### Definition 5.1

An exterior angle of a triangle is the angle at the vertex formed by one side of the triangle and an extension of the adjacent side of the triangle.

In the figure below, if D, A, C and E are points on the same line and C, B and F are points on the same line, then ∠DAB, ∠ECB and ∠FBA are exterior angles of triangle △ABC.




### Activity 5.2

1 Construct a triangle as in the Figure 5.14

2 Carefully cut ∠BAC and ∠ABC and join these two angles adjacently;

3 Put the two angles formed in (2) on the exterior angle at ∠C;

4 What relation do you observe?

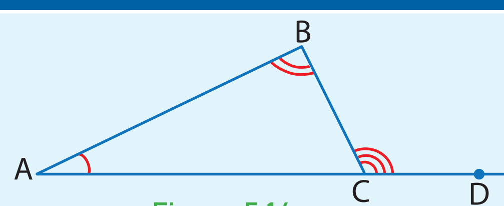



Figure 5.14

From your responses in Activity 5.2, observe that the sum of the measures of two angles of a triangle is equal to the measure the remote exterior angle.

### Theorem 5.4

An exterior angle of a triangle is equal to the sum of the two-remote interior angles. That is, in Figure 5.15, $m(\angle BCD) = m(\angle ABC) + m(\angle BAC)$.

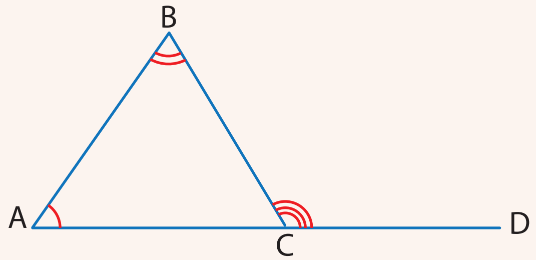



Figure 5.15

### Proof:

Given ∠ABC and ∠BAC are the two remote interior angles of the exterior angle ∠BCD of △ABC as in Figure 5.15, then we want to show that $m(\angle BCD) = m(\angle ABC) + m(\angle BAC)$

| | Statements | Reasons |
|---|---|---|
| 1 | $m(\angle ACB) + m(\angle BCD) = m(\angle ABC) + m(\angle ACB) + m(\angle BAC)$ | Both measure 180° |
| 2 | $m(\angle BCD) = m(\angle ABC) + m(\angle BAC)$ | Subtract $m(\angle ACB)$ from both sides |

Therefore, the measure of an exterior angle of a triangle is equal to the sum of the two-remote interior angles of the triangle.

## Example 5.7

An exterior angle of a triangle measures 120°. If the two remote interior angles of the triangle are congruent, then find the measure of each interior angle of the triangle.

### Solution

Let θ be the measure of one of the remote interior angles of the triangle.

Then,

$\theta + \theta = 120^\circ$ (By Theorem 5.4)

$\theta = 60^\circ$

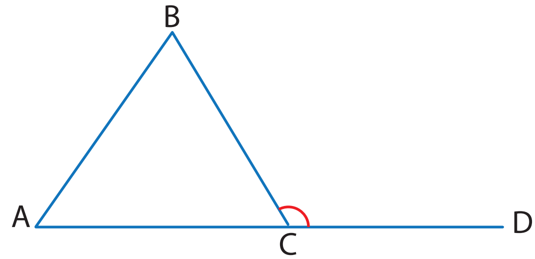
If $x$ is the measure of the third angle, then by the angle sum theorem of a triangle, $x + 60^\circ + 60^\circ = 180^\circ$.

Then, $x + 120^\circ = 180^\circ$ and $x = 180^\circ - 120^\circ = 60^\circ$

Therefore, the measure of each interior angle of the triangle is 60°.

## Example 5.8

Find the measure of an exterior angle of an equilateral triangle at any one of the vertices.




**Solution**

Let $\triangle ABC$ be equilateral triangle. Now, extend $\overline{AC}$ to D to get exterior angle $\angle BCD$ of $\triangle ABC$.

Then

$m(\angle CAB) + m(\angle ABC) + m(\angle BCA) = 180^0$ (Angle sum theorem)

$m(\angle BCA) + m(\angle BCD) = 180^0$ (straight angle)

But, $m(\angle CAB) = m(\angle ABC) = m(\angle BCA) = 60^0$ ($\triangle ABC$ is an equilateral triangle)

Hence, $m(\angle BCD) = 180^0 - m(\angle BCA) = 180^0 - 60^0 = 120^0$

Therefore, the measure of each exterior angle of an equilateral triangle is $120^o$.

## Exercise 5.2

1. If the exterior and interior angles of a triangle at one vertex are equal, then identify the type of the triangle
2. Two angles of a triangle measures $80^0$ and $50^0$ respectively. Find the measure of the exterior angle adjacent to the third angle.
3. If $\angle ABC$ and $\angle ECD$ are straight angles, then find the measure of the interior and exterior angles numbered 1, 2, 3 and 4 in Figure 5.16.

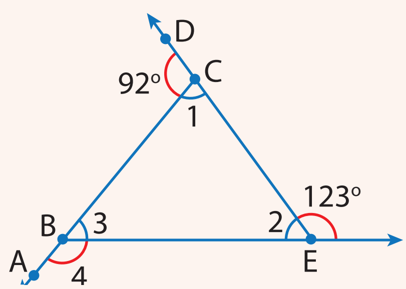



Figure 5.16

## 5.3 Theorems on Right-Angled Triangles

A right-angled triangle is a triangle having one right angle. The sides forming a right angle are perpendicular to each other.

The adjacent sides to the right angle of a right angled triangle are called legs and the side opposite to the right angle is called hypotenuse of the right - angled triangle.

In Figure 5.17, △ABC is right-angled triangle with m(∠ACB) = $90^0$. The sides $\overline{AC}$ and $\overline{BC}$ are legs and side $\overline{AB}$ is the hypotenuse.

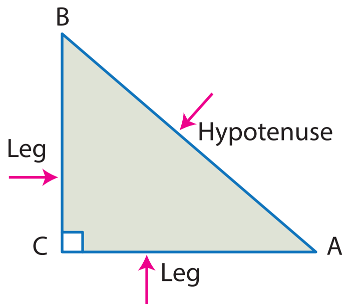



Figure 5.17

### 5.3.1 Euclid's Theorem and its Converse

Euclid is often referred to as the "Father of Geometry". He wrote the most enduring mathematical work of all time, the Elements, 13 volumes work. The Elements cover plane geometry, arithmetic and number theory, irrational numbers, and solid geometry.

**Theorem 5.5 (Euclid's Theorem)**

In a right angled triangle with an altitude drawn to the hypotenuse, the square of the length of each leg is equal to the product of the hypotenuse and the length of the adjacent segment into which the altitude divides the hypotenuse.

That is, as in Figure 5.18 if the altitude $\overline{CD}$ is drawn from C to the hypotenuse $\overline{AB}$ of a right angled triangle, $\Delta ABC$, then the following results are true.

a (AC)$^2$ = (AD)(AB) (i.e. $b^2 = ec$)

b (BC)$^2$ = (BD)(BA) (i.e. $a^2 = fc$)

c (CD)$^2$ = (AD)(BD) (i.e. $h^2 = ef$)

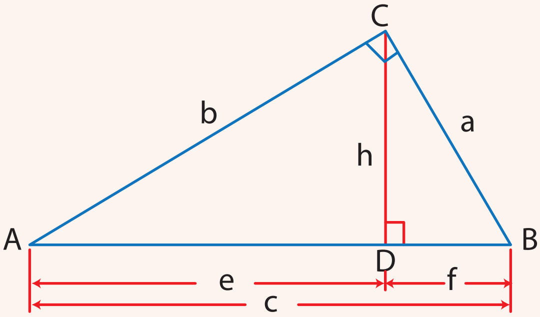



Figure 5.18 A right angled triangle

**Proof:**

| Statements | Reasons |
|---|---|
| 1 $m(\angle CAB) + m(\angle CBA) = m(\angle CAB) + m(\angle DCA)$ | ♦ Complementary angles |
| 2 $m(\angle CBA) = m(\angle DCA)$ | |
| 3 $\angle CAB$ | ♦ common angle |
| 4 $\Delta CDA \sim \Delta BCA$ | ♦ AA Similarity Theorem |
| 5 $\frac{CD}{BC} = \frac{DA}{CA} = \frac{CA}{BA}$ | ♦ Corresponding sides are proportional |
| 6 $\frac{h}{a} = \frac{e}{b} = \frac{b}{c}$ | ♦ From step 5 |
| 7 $b^2 = ce$ | |
| 8 $m(\angle CAB) + m(\angle CBD) = m(\angle CBD) + m(\angle BCD)$ | ♦ Complementary angles |
| 9 $m(\angle CAB) = m(\angle BCD)$ | |
| 10 $\angle CBD$ | ♦ common angle |
| 11 $\Delta CDB \sim \Delta ACB$ | ♦ AA Similarity Theorem |
| 12 $\frac{CD}{AC} = \frac{DB}{CB} = \frac{CB}{AB}$ | ♦ Corresponding sides are proportional |
| 13 $\frac{h}{b} = \frac{f}{a} = \frac{a}{c}$ | ♦ From step 12 |
| 14 $a^2 = cf$ | |
| Similarly, $\Delta ADC \sim \Delta CDB$ | ♦ AA Similarity Theorem |
| $h^2 = ef$ (this is called altitude theorem) | |

## Example 5.9

In the Figure 5.19 below, $\Delta ACB$ is right angled triangle with right angle at C and $\overline{CD}$ is altitude drawn from C to the hypotenuse. If $AD = 9cm$ and $DB = 6cm$, then find each of the following

a AC

b BC

c CD

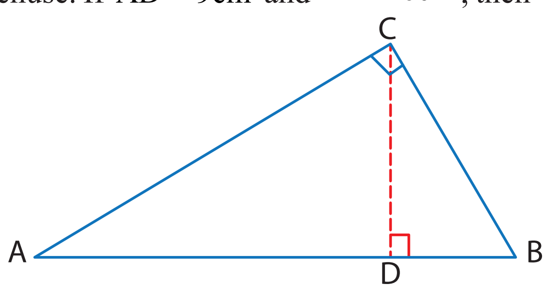



Figure 5.19

**Solution**

Note that $AB = AD + DB = 9cm + 6cm = 15cm$

Then by Euclid's Theorem

a $(AC)^2 = (AD)(AB) = (9cm)(15cm) = 135cm^2$

Thus, $AC = \sqrt{135cm^2} = 3\sqrt{15}cm$

b $(BC)^2 = (BD)(BA) = (6cm)(15cm) = 90cm^2$.

Thus, $BC = \sqrt{90cm^2} = 3\sqrt{10}cm$

c $(CD)^2 = (AD)(BD) = (9cm)(6cm) = 54cm^2$.

Therefore, $CD = \sqrt{54cm^2} = 3\sqrt{6}cm$

### Theorem 5.6 (Converse of Euclid's Theorem)

In a triangle if the square of each shorter side is equal to the product of the length of the longest side and the length of the adjacent segment into which the altitude to the longest side divides this side, then the triangle is a right angled triangle.

Given $\triangle ABC$, a right angled triangle with right angle at C as in Figure 5.20 and CD is an altitude draw from C to AB, if $(BC)^2 = (BD)(BA)$ and $(AC)^2 = (AD)(AB)$, then $\triangle ABC$ is a right angled triangle.

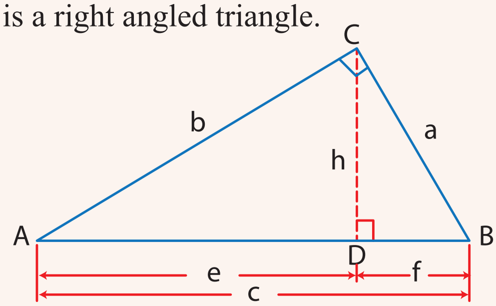


Figure 5.20

## Example 5.10

In Figure 5.21, prove that $\triangle$ OPQ is a right-angled triangle.

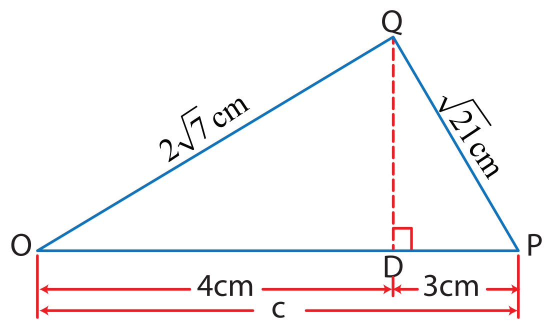


Figure 5.21

### Solution

From the given information

$$(OQ)^2 = \left(2\sqrt{7}\,cm\right)^2 = 28cm^2$$

$$(OD)(OP) = (4cm)\left[(4+3)cm\right] = (4cm)(7cm) = 28cm^2$$

Thus, we have $(OQ)^2 = (OD)(OP)$.

Similarly

$$(QP)^2 = 21cm^2 = (PD)(PO) = (3cm)(7cm)$$

Therefore, $\triangle OPQ$ satisfies the converse of Euclid's Theorem

Hence, $\triangle OPQ$ is right-angled triangle with right angle at Q.

## Exercise 5.3

1 In Figure 5.22, $\triangle ABC$ is a right-angled triangle and $\overline{CD}$ is the altitude to the hypotenuse. Find each of the following lengths.

a AC

b BC

c CD

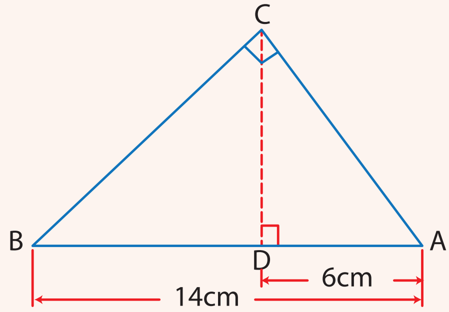
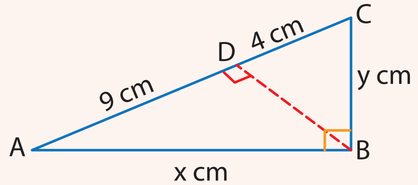



Figure 5.22

2 In $\Delta ACB$, if $\overline{CD} \perp \overline{AB}$, where D is a point on $\overline{AB}$ and AD = 3.6cm, AB = 10cm and CD = 4.8cm, then show that $\triangle ABC$ is right angled triangle.

3 In Figure 5.23 a) and Figure 5.23 b), $m(\angle ABC) = 90^0$, $\overline{BD}$ is the altitude to the hypotenuse $\overline{AC}$ of $\Delta ABC$. Find the values of $x$, $y$ and $z$



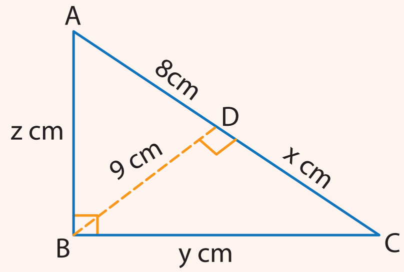




Figure 5.23

4 In Figure 5.24, $\Delta ABC$ is a right-angled triangle; CD is the altitude to the hypotenuse. Find the value of $x$, $y$ and $h$

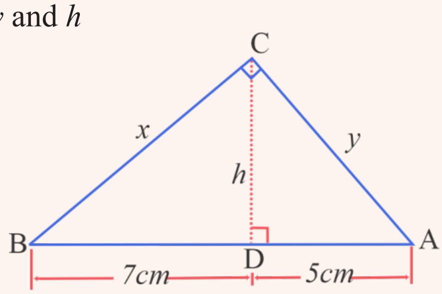
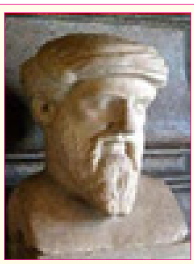



Figure 5.24

### 5.3.2 Pythagoras Theorem and its Converse

Early writers agree that Pythagoras was born in Samos, the Greek island in the eastern Aegean Sea. Pythagoras was a Greek religious leader and a philosopher who made developments in astronomy, mathematics and music theories.



## Activity 5.3

1 Add the first two formulas of the Euclid theorem and observe the result

2 In Figure 5.25, count the number of squares in C and the sum of squares in A and B. What relation do you observe between the length of hypotenuse and the lengths of the legs of the right angled triangle?

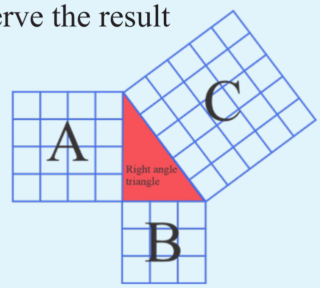


Figure 5.25

From your responses in Activity 5.3 observe that the sum of the squares of the two legs of a right angled triangle is equal to the square of its hypotenuse

### Theorem 5.7 (Pythagoras Theorem)

Given a right angled triangle with sides of lengths a, b and c, where c is the hypotenuse then

$$a^2 + b^2 = c^2$$

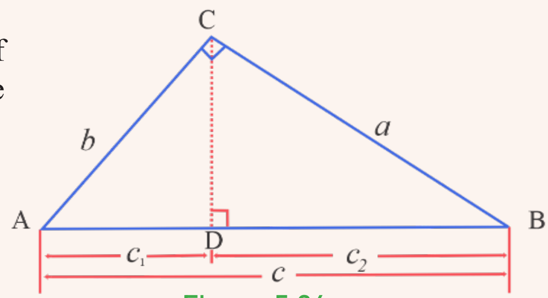


Figure 5.26

**Proof:**

Let ΔABC be right-angled triangle with right angle at C and $\overline{CD} \perp \overline{AB}$ as shown in Figure 5.26. Then we want to show that $a^2 + b^2 = c^2$

| | Statements | Reasons |
|---|---|---|
| 1 | $b^2 = c_1c$ | ♦ By Euclids' theorem |
| 2 | $a^2 = c_2c$ | ♦ By Euclids' theorem |
| 3 | $a^2 + b^2 = c_2\,c + c_1\,c = c\,(c_2 + c_1)$ | ♦ Add the two Euclids' formula in Steps 1 & 2 |
| 4 | $a^2 + b^2 = c^2$ | |

Therefore, for any right-angled triangle whose legs are of lengths *a* and *b* and hypotenuse c, you have $a^2 + b^2 = c^2$

## Example 5.11

If the length of the diagonal of a square is 20cm, then find the lengths of its sides.

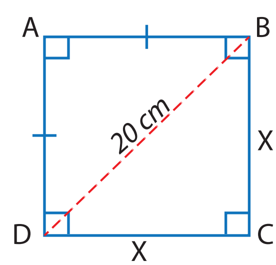


Figure 5.27

All the sides of a square are equal in length. The diagonal divides the square into two congruent right triangles. Consider square ABCD as shown in Figure 5.27. Let $x$ be the side of the square. Then

$(AB)^2 + (AD)^2 = (BD)^2$ (By pythagoras Theorem)

$x^2 + x^2 = (20cm)^2$

$2x^2 = 400cm^2$

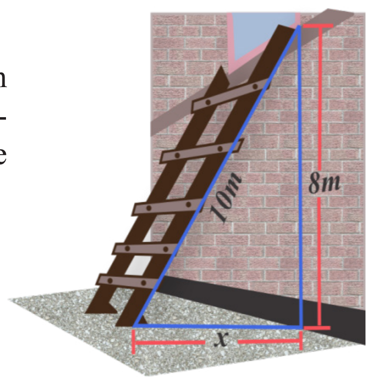
$x^2 = 200cm^2$

$x = \sqrt{200cm^2} = \sqrt{100 \times 2cm^2} = 10\sqrt{2}$ cm

## Example 5.12

A ladder of length 10m leans against a vertical wall. If the top of the ladder reaches 8m up the wall, find the distance from the wall to foot of the ladder.

### Solution




Assume that the wall is perpendicular to the ground. Then the ladder with the wall and the ground form a right-angled triangle. Let x be the distance from the wall to the foot of the ladder. Then, by Pythagoras theorem

$x^2 + (8m)^2 = (10m)^2$

$x^2 = 100m^2 - 64m^2 = 36m^2$

$x = \sqrt{36m^2} = 6m$

Therefore, the foot of the ladder is 6m away from the wall.

**Theorem 5.8 (Converse of Pythagoras Theorem)**

If the lengths of the sides of $\triangle ABC$ are $a$, $b$ and $c$ and $a^2 + b^2 = c^2$, then the triangle is right angled triangle and the right angle is opposite to the side of length c.

## Example 5.13

Is $\triangle ABC$ with $AB = 5m$, $BC = 4m$ and $CA = 3m$ a right angled triangle?

### Solution

The Square of the sides of the triangle are:

$(AB)^2 = (5m)^2 = 25m^2$

$(BC)^2 = (4m)^2 = 16m^2$

$(CA)^2 = (3m)^2 = 9m^2$

$(BC)^2 + (CA)^2 = 16m^2 + 9m^2 = 25m^2 = 25m^2 = (AB)^2$

Therefore, $\Delta ABC$ is right - angled triangle, by the coverse of Pythagoras Theoem.

## Exercise 5.4

1 Which of the following triples of numbers form lengths of sides of a right angle triangle?

a $2cm, 3cm, 4cm$

b $6cm, 6\sqrt{3}cm, 12cm$

c $2cm, 2\sqrt{5}cm, 4cm$

d $3cm, 5cm, 7cm$

e $9cm, 12cm, 15cm$

f $10cm, 24cm, 26cm$

2 If $AB = 4cm$ and $AC = 2\sqrt{3}cm$ are the lengths of two sides of a right angled triangle ΔABC, right angled at C, then find the length of the third side BC.

3 A carpenter builds a small tabletop with length 56 cm and width 33 cm for his costumer. If diagonal of the tabletop measures 60 cm, does the tabletop have right angle corners? Justify your answer?

4 In Figure 5.28,, ΔABC is a right angled triangle, with right angle at B, $BD \perp AC$, $BE = BC$, $BE = 6cm$ and $AC = 12cm$. Find

a AB

b DC

c BD

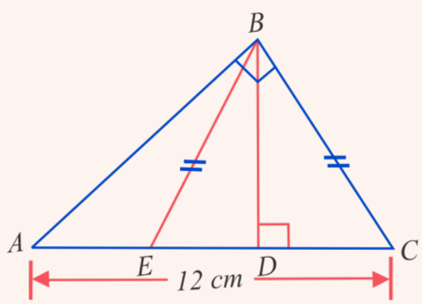



Figure 5.28

5 If a square shaped garden has a perimeter of 360m, then find the

a length of the side of the garden.

b diagonal of the garden.

6 A door has 3 meter height and 1 meter width. If a square piece of play wood has side length of 2.2 meter, can the play wood inter through the door? How do you know? Show your work with a figure.

7 (Project work): Draw any triangle ABC and fold the three sides of the triangle to get the mid points of the sides, next draw perpendicular lines from side AB to side BC and from side AC to BC. Fold inwards ΔADE at line segment DE, ΔDBF at line segment DF and ΔCEG at line segment EG. What can you say about the sum of the interior angles of the triangle?

# Unit Summary

1 The sum of the measures of interior angles of any triangle is $180^0$.

2 Adjacent angles have a common vertex and a common side but not common interior point.

3 Adjacent supplementary angles form a straight angle.

4 Adjacent complementary angles form a right angle.

5 Any of the three sides of a triangle can be considered as base of the triangle.

6 Triangles can be classified based on their length of sides and angle measures.

7 Every equilateral triangle is isosceles, but the converse is not true.

8 Every equilateral triangle is acute angled triangle.

9 For any ΔABC, $\overline{AB} \cong \overline{AC}$ if and only if $\angle ABC \cong \angle ACB$

10 Let ΔABC be a triangle such that $CD \perp AB$, where $D$ is a point on $AB$, $AB = c$, $BC = a$, $AC = b$, $AD = e$ and $DB = f$, then

a ΔABC is a right-angled triangle if and only if $a^2 = fc$ and $b^2 = ec$

b ΔABC is a right-angled triangle if and only if $a^2 + b^2 = c^2$.

## Review Exercises

1 Which of the following set of numbers could not be the length of sides of a right-angled triangle?

a 0.75cm, 1cm, 1.25cm

b 1mm, $\frac{3}{2}mm$ , 2mm

c 6mm, 8mm, 10mm

d 5m, 12m, 13m

2 From the information given in Figure 5.29, find the value of x.

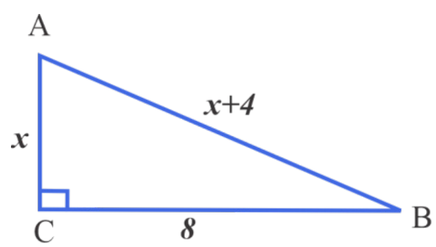



Figure 5.29

3 If the length of the diagonal of a square is 400cm, then find the length of each side of the square.

4 How long is the height of an equilateral triangle of side length 10cm?

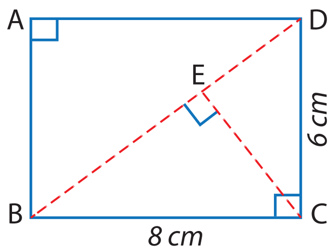
5 One leg of an isosceles right angled triangle is 3cm long. What is the length of the hypotenuse?

6 Suppose △ABC is a right angled triangle with hypotenuse $\overline{AB}$ and altitude $\overline{CD}$ to $\overline{AB}$.

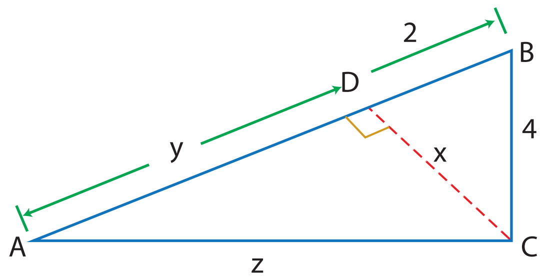
a If $AD = 4\text{cm}$, $CD = 2\text{cm}$, then find the lengths of $\overline{BC}$ and $\overline{DB}$.

b If $AC = BC = 3\text{cm}$, then find the lengths of $\overline{AD}$, $\overline{BD}$ and $\overline{CD}$.

7 In the figure below find the lengths of $\overline{ED}$ and $\overline{CE}$.




8 In the figure below ΔABC is aright angled triangle with right angle at C and $\overline{CD} \perp \overline{AB}$, find the values of $x$, $y$ and $z$.

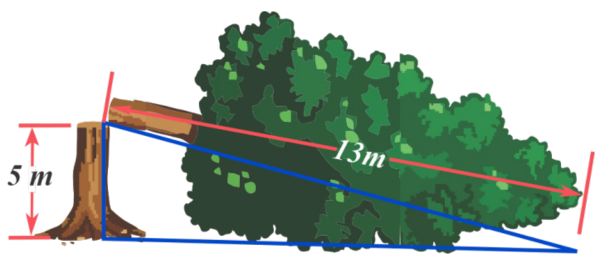



9 A 6m long pole is used to support an electric pole. The pole is anchored at 4m far from the base of the electric pole. What is the height the electric pole?

10 A farmer has a rectangular shaped farm land. The length of one side is 25m and the diagonal is 35m. If in each square meter there is one plant, how many plants does the farmer have in this farm land?

11 A verical electric pole casts a shadow of 24 meters long. If the tip of the shadow is 25 meters away from the top of the pole, how high is the pole from the ground?

12 A tree of 18 meters height is broken off 5 meters from the ground as shown in the figure below. How far from the foot of the tree will the top strike the ground?



# Unit 6

# LINES AND ANGLES IN A CIRCLE

## Learning outcomes:

**After completing this unit, you will be able to:**

- understand circles and its parts;
- identify the relationship between lines and circles;
- determine measures of central & inscribed angles and angles formed by intersecting chords;
- apply the concept of lines and angles in a circle to solve real life problems.

## *Key terms*

- chord
- secant line
- inscribed angle
- major arc
- semi-circle
- tangent line
- central angle
- arc
- minor arc
- circle
- segment
- sector
- minor segment
- major segment
- miner sector
- major sector

## Introduction

The concept of circle is important in our day-to-day activities and it is applicable in many fields. In your previous grades, you have learnt about circle and its properties. In this unit, you will learn about the relation between circle and lines; tangent lines, secant lines, chords and angles formed by intersecting chords.

## 6.1 Circles and Lines

In this section, you will learn about circles and parts of a circle (semi-circle, minor arc and major arc), tangent lines, secant lines and chords of a circle.

### Definition 6.1

The set of all points in a plane that are at equal distance (or equidistant) from a fixed point is called a circle; the fixed point is called the center of the circle and the distance from the center of the circle to any point on the circle is called the radius of the circle.

Circles are named by their centers. For example, in Figure 6.1, O is the center of the circle and P is a point on the circle. Then OP is the radius of the circle and the circle is named as circle O.

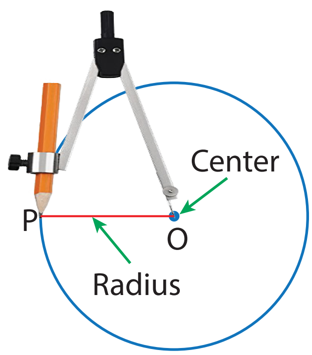



Figure 6.1 A circle with center O and radius OP

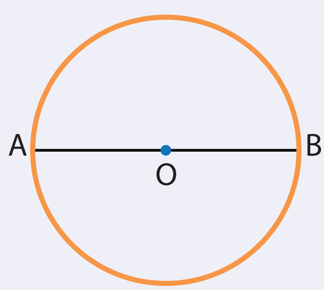
## Note

1 We use the word radius for both the distance from the center to any point on the circle and the line segment.

2 Given a circle, the distance around the circle is called circumference of the circle.

3 A segment through the center of a circle with end points on the circle is called a diameter of the circle.

In the figure to the right $\overline{AB}$ is a segment through the center of the circle and end points on the circle.




4 The word diameter is used for both the distance and the line segment

## Definition 6.2

Given a circle, any part of the circle is called an **arc** of the circle.

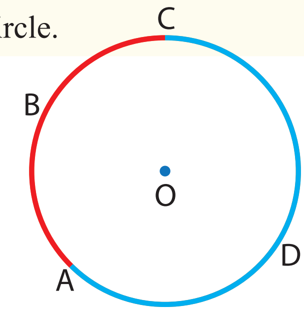



Figure 6.2 Arcs of a circle

In Figure 6.2, the curve on circle O from A to C through B is an arc of the circle, denoted by $\overset{\frown}{ABC}$ and read as arc ABC. The curve on circle O from A to C through D is also another arc of circle O, denoted by $\overset{\frown}{ADC}$ read as arc ADC.

### Classifications of Arcs

Arcs are classified as semi-circle, minor arc and major arc based on their sizes as compared to the circle.

a An arc which is half of a circle is called a **Semi-circle.**

In Figure 6.3, if O is the center of the circle and $\overline{AB}$ is a diameter of the circle, then the arcs $\overset{\frown}{ACB}$ and $\overset{\frown}{ADB}$ are both semi-circles "semi" means "half"

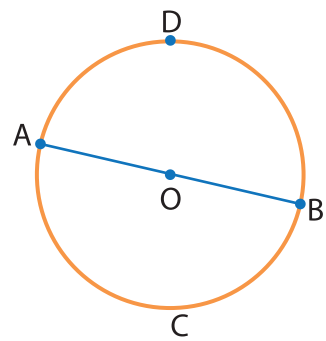



Figure 6.3 A Circle

b Part of a circle which is smaller than a semi-circle is called a **minor arc** and a part of a circle which is larger than a circle is called a **major arc.**

In Figure 6.4, O is the center of the circle, $\overline{AB}$ a diameter of the circle and C, D and E are points on the circle. Then

i the arcs $\overset{\frown}{ADE}$ and $\overset{\frown}{DEB}$ are both minor arcs;

ii the arcs $\overset{\frown}{AEC}$, $\overset{\frown}{ACE}$ and $\overset{\frown}{ACD}$ are all major arcs.

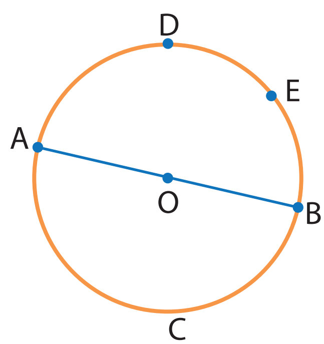



Figure 6.4 minor and major arcs

### Tangent lines, Secant Lines and chords of a circle

## Definition 6.3

Given a circle and a line in the same plane:

a if the line touches the circle only at one point, then the line is called a **tangent line** and the point at which the tangent line touches the circle is called **point of tangency**;

b if the line intersects the circle at two points, then the line is called a **secant line**.

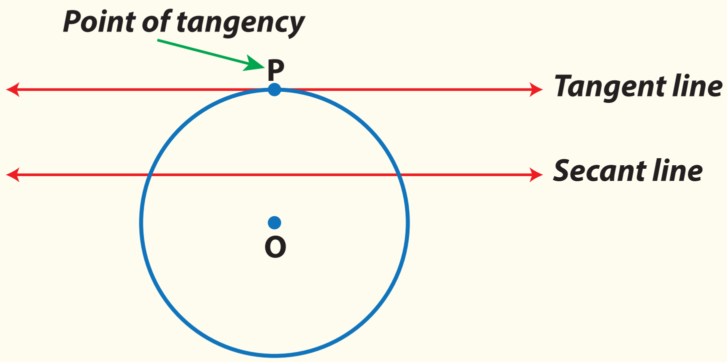



Figure 6.5 Tangent and secant lines

c A line segment joining any two points of the circle is called a chord.

d A diametre of the circle is twice its radius.

## Activity 6.1

Do the following activity in a group of two or three students

Use ruler and compass to do the following tasks.

1. Draw a circle of radius of 2cm and locate its center, say O.
2. Draw at least two line segments through the center of the circle with ends points on the circle.
3. Draw at least three line segments that do not passes through the center of the circle with points on the circle.
4. Measure the line segments in 2 and 3 above and compare their lengths.

From your responses in Activity 6.1, observe that the line segment through the center are longer in size than the line segment not through the center. Thus, diameter is the longest chord.

## Example 6.1

Consider a circle O of radius 5cm. Which one of the following can be a possible length of a chord of the circle?

a 6cm b 10cm c 14cm

### Solution

The radius of the circle is 5cm, hence its diameter is 10cm.

Therefore, the longest chord in the circle has length 10cm and any chord in the circle has length less than or equal to 10 cm.

So, you can have a chord of length 6cm and you can also have a chord of length 10 cm. But it is not possible to have a chord of length 14cm.

## Activity 6.2

Do the following activity in a group of two or three students

Use ruler and compass to do the following tasks.

1. Draw a circle of radius 3cm and locate its center, say O.
2. Draw at a line tangent to the circle at a point on the circle.
3. Draw a radius from the center of the circle to the point of tangency in 2 above.
4. Measure the angle that the radius makes with the tangent line.

From your responses in Activity 6.2, observe that the radius of a circle is perpendicular to the tangent line to the circle at the point of tangency.

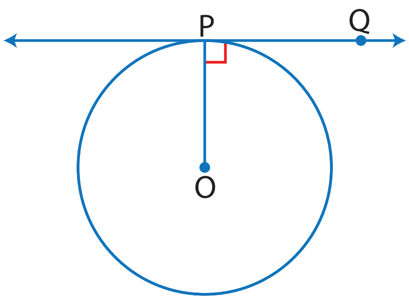



Figure 6.6 A circle and a tangent line

In Figure 6.6, O is the center of the circle, $\overline{OP}$ in a radius of the circle, $\overleftrightarrow{PQ}$ is a line tangent to the circle and P is the point of tangency.

$\angle OPQ$ is a right angle, an angle that the radius $\overline{OP}$ makes with the tangent line $\overleftrightarrow{PQ}$

## Example 6.2

In the Figure 6.7, C is the center of the circle, $\overleftrightarrow{AD}$ and $\overleftrightarrow{AB}$ are tangent lines to circle C at D and B respectively. If the radius of the circle is 5cm and AC = 13cm, then find the lengths of the line segments $\overline{AD}$ and $\overline{AB}$.

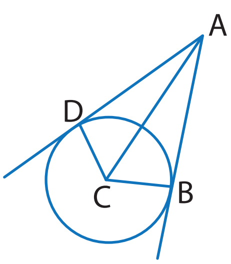



Figure 6.7 Circle C

### Solution

From the given information, observe that $\overline{CD} \perp \overline{AD}$ and $\overline{CB} \perp \overline{AB}$.

Then, $\Delta ABC$ and $\Delta ADC$ are right angled triangles.

a By Pythagoras Theorem, $(AD)^2 + (CD)^2 = (AC)^2$.

$(AD)^2 + (5cm)^2 = (13cm)^2$

$(AD)^2 + 25cm^2 = 169cm^2$

$(AD)^2 = 169cm^2 - 25cm^2 = 144cm^2$.

$AD = \sqrt{144cm^2} = 12cm$

b  Again by Pythagoras Theorem, $(AB)^2 + (CB)^2 = (AC)^2$.

$(AB)^2 + (5cm)^2 = (13cm)^2$

$(AB)^2 + 25cm^2 = 169cm^2$

$(AB)^2 = 169cm^2 - 25cm^2 = 144cm^2$.

$AB = \sqrt{144cm^2} = 12cm$

## Note

1 If the diameter bisects a chord of a circle, then it is perpendicular to the chord of the circle.

2 Arcs intercepted by parallel lines have equal measure.

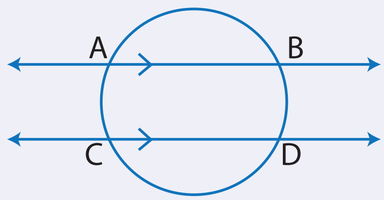



Figure 6.8

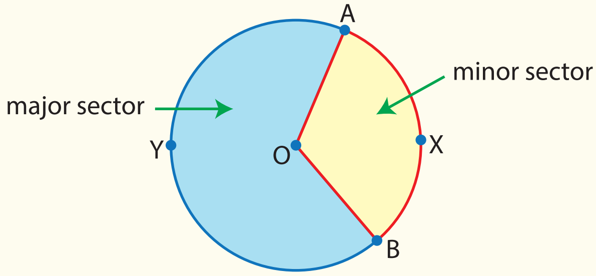
In Figure 6.8 $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, then $m(\overset{\frown}{AC}) = m(\overset{\frown}{BD})$

# Sectors and Segments of Circles

## Sectors of a Circle

### Definition 6.4

Consider a circle with center O as shown in the figure below. If A, X, B and Y are points on the circle, then the region bounded by $\overset{\frown}{AXB}$ and the two radii $\overline{OA}$ and $\overline{OB}$ is called a sector of the circle.



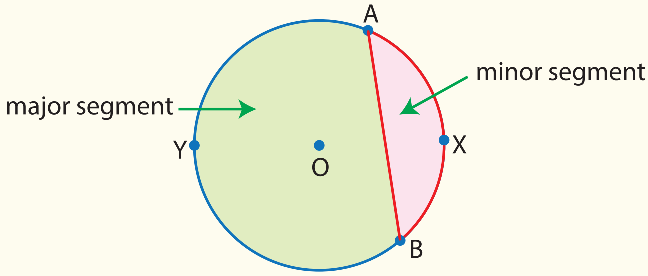

a  If $\overset{\frown}{AYB}$ is a major arc, then the sector bounded by $\overset{\frown}{AYB}$ and the two radii $\overline{OA}$ and $\overline{OB}$ is called a major sector.

b  If $\overset{\frown}{AXB}$ is a minor arc, then the sector bounded by $\overset{\frown}{AXB}$ and the two radii $\overline{OA}$ and $\overline{OB}$ is called a minor sector.

## Segment of a circle

### Definition 6.5

Consider a circle with center O as shown in the figure below. If A, X, B and Y are points on the circle, then the region bounded by $\overset{\frown}{AXB}$ and the chord $\overline{AB}$ is called a segment of the circle.




a If $\overset{\frown}{AYB}$ is a major arc, then the sector bounded by $\overset{\frown}{AYB}$ and the chord $\overline{AB}$ is called a major segment.

b If $\overset{\frown}{AXB}$ is a minor arc, then the sector bounded by $\overset{\frown}{AXB}$ and the chord $\overline{AB}$ is called a minor segment.

## Exercise 6.1

1 In Figure 6.9, O is the center of the circle, A, B, C and P are points on the circle and $\overline{BC}$ passes through O. Identify each the following.

a a diameter
b radii
c chords
d a tangent line
e a secant line
f major arcs
g minor arcs
h semi-circles
i a sector

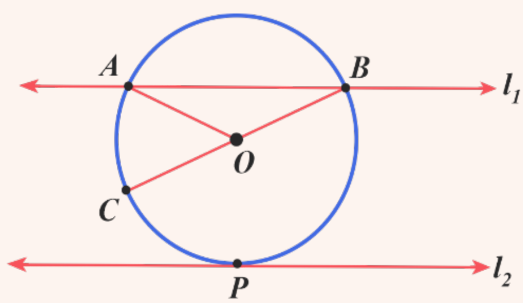



Figure 6.9

2 Consider a circle with center O and a radius of 5cm. Then

a how many chords can you draw?

b how many diameters can you have?

3 In Figure 6.10 to the right, O is the center of the circle

a Which parts of a circle (labeled by R's) are segments? Which are sectors?

b Which of the following combinations form a segment? Which of these combination form a sector?

i The region covered by $R_1$, $R_5$ and $R_6$

ii The region covered by $R_1$ and $R_6$

iii The region covered by $R_1$, $R_2$, $R_5$ and $R_6$

iv The region covered by $R_3$ and $R_4$

c Identify minor and major sectors

d Identify minor and major segments

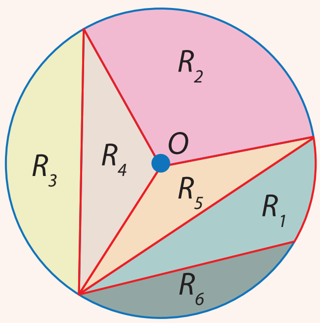



Figure 6.10

## 6.2 Central Angles and Inscribed Angles of circles

In the previous sub-unit, you have learnt about types of arcs, chords, segments and sectors and in this sub-unit, you will learn the relationships betweeen arcs, central angles and inscribed angles on a circle.

### Definition 6.6

Given a circle with center O and radius r:

a an angle whose vertex is at the center of the circle and whose sides are two radii of the circle is called a **central angle**;

b an angle whose vertex is on the circle and whose sides are two chords of the circle is called an **inscribed angle.**

In Figure 6.11, O is the center of the circle, A, C, B and P are points on the circle and $\overline{AP}$ and $\overline{BP}$ are chords of the circle.

a ∠AOB is central angle subtended by arc ACB

b ∠APB is an inscribed angle subtended by arc ACB;

c Arc ACB is called an arc intercepted by ∠AOB and

d Arc ACB is intercepted by ∠APB

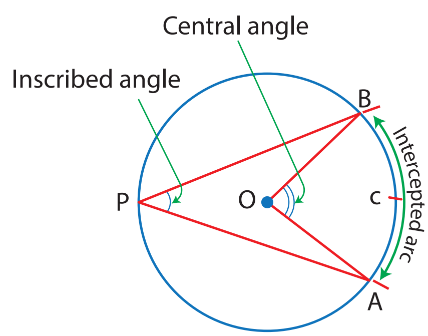



Figure 6.11 Central and inscribed angles of a circle

## Note

In Figure 6.11 above

a $\angle AOB$ is a central angle of circle O. In this case, we say that

- arc $\overset{\frown}{ACB}$ subtends $\angle AOB$ and
- $\angle AOB$ intercepts arc $\overset{\frown}{ACB}$

b $\angle APB$ is an inscribed angle of circle O. In this case, we say that:

- arc $\overset{\frown}{ACB}$ subtends $\angle APB$ and
- $\angle APB$ intercepts arc $\overset{\frown}{ACB}$

### Theorem 6.1

In a given circle, the meaure of a central angle is equal to the measure of its intercepted arc.

In Figure 6.12, O is the center of the circle and A, B and C are points on the circle. Arc ACB is the arc intercepted by the central angle $\angle AOB$ and $m(\angle AOB) = m(\overset{\frown}{ACB})$.

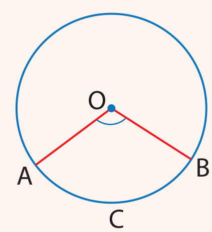



Figure 6.12 A circle with center O

## Example 6.3

In Figure 6.13, O is the center of the circle, A, X, B, Y, C ,Y and Z are points on the circle, $\overline{AC}$ is a diameter of the circle and $m(\angle AOB) = 80°$.

Then find

a $m(\overset{\frown}{AXB})$

b $m(\overset{\frown}{BYC})$

c $m(\overset{\frown}{AZC})$

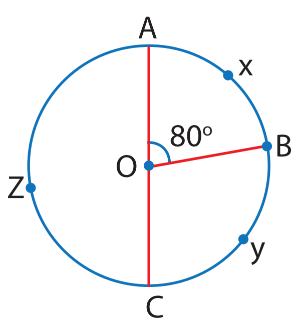



Figure 6.13 Circle

### Solution

a By Theorem 6.1, $m(\overset{\frown}{AXB}) = m(\angle AOB) = 80°$.

b As $\overline{AC}$ is a diameter of the circle, ∠AOC is a straight angle. That is, $m(\angle AOC) = 180°$.

Thus, $m(\angle AOB) + m(\angle BOC) = 180°$.

This implies $m(\angle BOC) = 180° - m(\angle AOB) = 180° - 80° = 100°$.

Then, by Theorem 6.1, $m\left(\widehat{BYC}\right) = m(\angle BOC) = 100°$.

c Using Theorem 6.1, $m\left(\widehat{AZC}\right) = m(\angle AOC) = 180°$.

## Example 6.4

In Figure 6.14, O is the center of the circle and $\overline{AC}$ and $\overline{BD}$ are diameters of the circle. Then find. $m\left(\widehat{AB}\right)$, $m\left(\widehat{BC}\right)$ and $m\left(\widehat{CD}\right)$

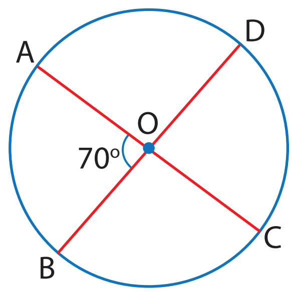


Figure 6.14

### Solution

As O is the center of the circle, ∠AOB, ∠BOC, ∠COD and ∠AOD are central angles.

a $m\left(\widehat{AB}\right) = m(\angle AOB) = 70^0$ (by Theorem 6.1)

b $m\left(\widehat{BC}\right) = m(\angle BOC) = 180^0 - 70^0 = 110^0$ (because $m(\angle AOC) = 180°$)

c $m\left(\widehat{CD}\right) = m(\angle COD) = 180^0 - 110^0 = 70^0$ ($m(\angle BOD) = 180°$).

### Measures of Inscribed Angels and Intercepted Arcs

## Activity 6.3

Using a ruler and a protractor:

a draw a circle of radius 2cm with center O;

b draw a central angle and an inscribed angle which are subtended by the same arc on the circle;

c measure the two angles in (b).

d what relation do you get between the measures of the two angles?

For your responses on Activity 6.3, observe that the measure of the inscribed angle is half of the measure of the central angle that are subtended by the same arc.

In Theorem 6.1, the measure of a central angle and the measure of the arc intercepted by the central angle are equal.

In the following Theorem, a relationship between the measure of an inscribed angle and the measure of the arc intercepted by the inscribed angle is given.

## Theorem 6.2

In any circle, the measure of an inscribed angle is half of the measure of the arc intercepted by the inscribed angle.

In Figure 6.15, if O is the center of the circle and P, A,C and B are points on the circle, then m $(\angle APB) = \frac{1}{2} m\left(\overset{\frown}{ABC}\right)$

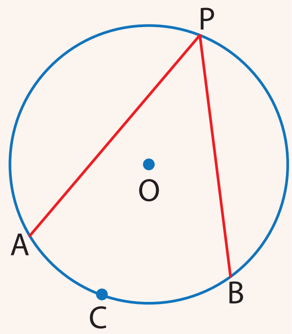


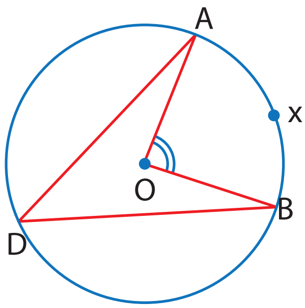

Figure 6.15 Circle O

Using the results in Theorem 6.1 and Theorem 6.2, given a circle O, if A, X, B and D are points on the circle, as shown in the figure below, then $\angle ADB$ is an angle inscribed angle subtended by $\overset{\frown}{AXB}$. Then

a $m(\angle ADB) = \frac{1}{2} m\left(\angle AOB\right) = \frac{1}{2} m(\overset{\frown}{AXB})$

b $m\left(\angle AOB\right) = 2m\left(\angle ADB\right)$

c $m\left(\angle AOB\right) = m(\overset{\frown}{AXB})$




## Note

In a circle;

a the measures of inscribed angles which are subtended by the same arc are equal;

b the measure of an inscribed angle is half of the measure of a central angle that is subtended by the same arc.

## Example 6.5

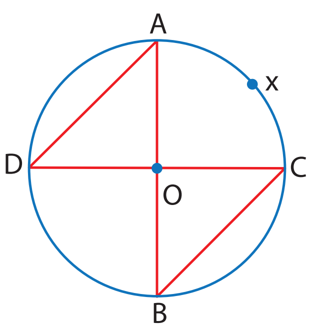
In the figure below, O is the center of the circle, $\overline{AB}$ and $\overline{CD}$ are diameters of the circle and $m(\overset{\frown}{AXC}) = 90^0$. Then evaluate each of the following.

a $m(\angle AOC)$

b $m(\angle ADC)$

c $m(\angle ABC)$

d $m(\angle DCB)$



### Solution

From the given information, ∠AOC is a centeral angle and ∠ADC, ∠ABC and ∠DCB are inscribed angles on the circle.

a $m(\angle AOC) = m(\overset{\frown}{AXC}) = 90^0$ (by Theorem 6.1)

b $m(\angle ADC) = \frac{1}{2}\ m(\overset{\frown}{AXC}) = \frac{1}{2}\left(90^0\right) = 45^o$ (by Theorem 6.2)

c $m(\angle ABC) = \frac{1}{2}\ m(\overset{\frown}{AXC}) = \frac{1}{2}\left(90^0\right) = 45^o$ (by Theorem 6.2)

d $m(\angle DOB) = 90^0$ (vertically opposite angles are congruent)

Then, $m(\angle DCB) = \frac{1}{2}m\left(\overset{\frown}{BD}\right) = \frac{1}{2}m(\angle DOB) = \frac{1}{2}(90^o) = 45^o$ (by Theorem 6.2)

Therfore, $m(\angle DCB) = 45^0$.

## Example 6.6

In Figure 6.16, O is the center of the circle and $\overline{AC}$ is diameter of the circle. If $y = 112^0$, then find the value of $x$.

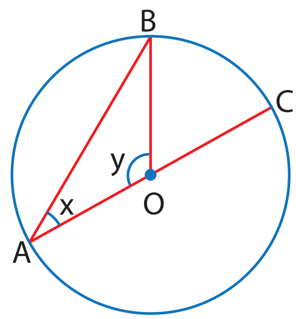


Figure 6.16

### Solution

In Figure 6.16, ∠BOC is a central angle and ∠BAC is an inscribed angle.

$m(\angle BOC) = 180^0 - y = 180^0 - 112^0 = 68^0$ (∠AOC is straight angle).

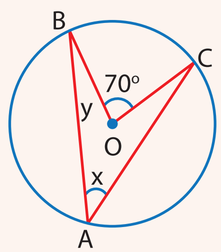
$m(\angle BAC) = \frac{1}{2}m(\angle BOC) = \frac{1}{2}(68^o) = 34^o$ (by Theorem 6.2)

This implies, $x = 34^o$.

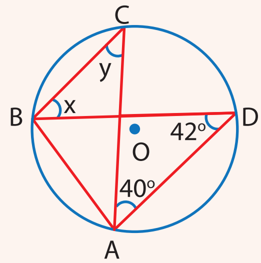
## Exercise 6.2

1 For each of th figures below, if O is the center of the circle and points A,B,C and D are points on the circle, then find the measures of the angles indicated by $x$ and $y$.

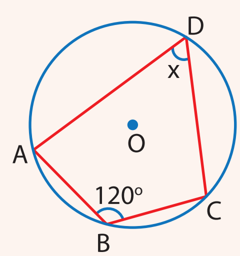
a




b

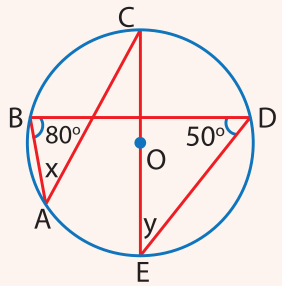



c



2 What is the measure of an inscribed angle subtended by a semi-circle?

3 In the figure below, if O is the center of the circle, $\overline{EC}$ is a diameter of the circle and A, B, C, D and E are points on the circle, then find the values of $x$ and $y$.

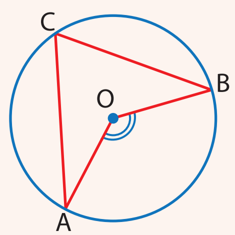



4 If the measure of a central angle of a circle of radius 3cm is $90^0$, then

a find the length of the chord intercepted by the central angle;

b find the measure of any inscribed angle intercepted by the same arc as the given central angle.

5 In the figures below, O is the center of the circle. If $m(\overset{\frown}{ACB}) = 212°$, then find m(∠AOB) and m(∠ACB).



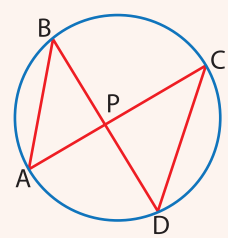

6 In the figure below A,B,C and D are points on the circle and P is any point inside the circle which is the intersection of chords $\overline{AC}$ and $\overline{BD}$. If m(∠CPD) = $120^0$, m(∠PCD) = $30^0$. Then

a Are m(∠BAC) and m(∠BDC) equal?

b Are m(∠ABD) and m(∠ACD) equal?

c Find m(∠BAC) and m(∠BPA).

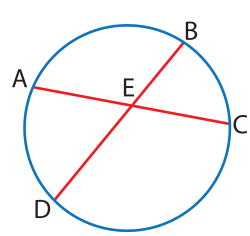
d Is P the center of the circle? Why?




## 6.3 Angles Formed by Two Intersecting Chords in a Circle

When two chords intersect inside a circle, the circle is divided into four arcs and there are four angles that are formed in the process.

In the figure on the right, A, B, C and D are points on the circle and $\overline{AC}$ and $\overline{BD}$ are cords of the circle.



## Activity 6.4

Using ruler and protractor,

a draw a circle of radius 2cm with center O;

b draw two chord $\overline{AC}$ and $\overline{BD}$ that are intersecting at point E inside the circle

c measure the angles ∠AED, ∠ACD and ∠BDC;

d compare m(∠AED) and $\frac{1}{2}\left(m(\angle ACD) + m(\angle BDC)\right)$.

For your responses in Activity 6.4, observe that the measure of an angle formed by two chords is half of the sum of the measure of the arcs inscribed it and its vertically opposite angle.

### Theorem 6.3

The measure of an angle formed by two chords intersecting inside a circle is half of the sum of the measures of the arcs subtending the angle and its vertically opposite angle.

That is, if $\overline{AB}$ and $\overline{CD}$ are two intersecting chords of a circle at point P, as in Figure 6.17, then

$$m(\angle BPD) = \frac{1}{2}\left[m(\overset{\frown}{BXD}) + m(\overset{\frown}{AYC})\right]$$

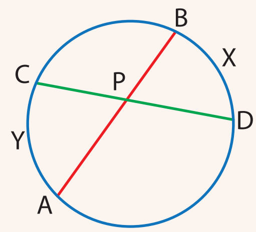



Figure 6.17

**Proof:**

Consider Figure 6.18.

a $m(\angle BPD) + m(\angle APD) = m(\angle BAD) + m(\angle CDA) + m(\angle APD)$ (both sums are $180^0$)

This implies, $m(\angle BPD) = m(\angle BAD) + m(\angle CDA)$.

b $m(\angle BAD) = \frac{1}{2}m(\overset{\frown}{BXD})$ (by Theorem 6.2) $m(\angle CDA) = \frac{1}{2}m(\overset{\frown}{AYC})$ (by Theorem 6.2)

c $m(\angle BPD) = \frac{1}{2}m(\overset{\frown}{BXD}) + \frac{1}{2}m(\overset{\frown}{AYC})$ ( from Steps a and b)

Therefore, $m(\angle BPD) = \frac{1}{2}\left[m(\overset{\frown}{BXD}) + m(\overset{\frown}{AYC})\right]$.

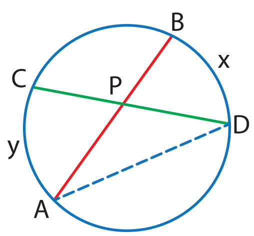



Figure 6.18

## Example 6.7

In Figure 6.19, $\overline{AC}$ and $\overline{BD}$ are chords of the circle that intersect at O. If $m(\overset{\frown}{AB}) = 82^0$ and $m(\overset{\frown}{DC}) = 46^0$, then find the value of β.

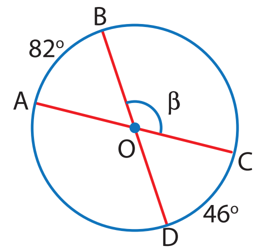



Figure 6.19

### Solution

$\overline{AC}$ and $\overline{BD}$ are chords of the circle that intersect at O.

Then

$$m(\angle AOB) = \frac{1}{2}\left(m(\overset{\frown}{DC}) + m(\overset{\frown}{AB})\right) \text{ (by Theorem 6.3)}$$

$$= \frac{1}{2}\left(82^0 + 46^0\right) = 64^0$$

This implies, $m(\angle AOB) = 64^0$

$\beta = m(\angle BOC) = 180^0 - m(\angle AOB)$ ($\angle AOC$ is a straight angle).

$=180° - 64° = 116°$

## Example 6.8

In Figure 6.20, $\overline{AB}$ and $\overline{DC}$ are parallel chords of circle O, $\overline{AC}$ and $\overline{BD}$ are chords of the circle that intersect at O and $m(\angle ABD) = 33^0$. Find m (∠AOD) and m (∠ACD).

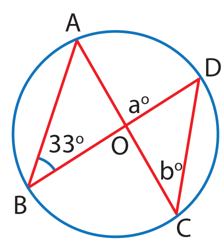



Figure 6.20

### Solution

$\overline{AC}$ and $\overline{BD}$ are chords that intersect at point O inside the circle.

a $m(\overset{\frown}{AD}) = 2m(\angle ABD) = 2 \times 33^0 = 66^0$ (by Theorem 6.2)

b $m(\angle BDC) = 33^0$ ($\angle BDC$ and $\angle ABD$ are alternate interior angles)

c $m(\overset{\frown}{BC}) = 2m(\angle BDC) = 2 \times 33^0 = 66^0$ (by Theorem 6.2)

d $m(\angle ACD) = \frac{1}{2} m(\overset{\frown}{AD}) = \frac{1}{2}(66^0) = 33^0$ (by Theorem 6.2)

e $m(\angle AOD) = \frac{1}{2}\left[m(\overset{\frown}{AD}) + m(\overset{\frown}{BC})\right] = \frac{1}{2}\left[66^0 + 66^0\right] = 66^0$ (by Theorem 6.3)

## Note

1 An inscribed angle on a circle subtended by a diameter is $90^0$.

In Figure 6.21, if $\overline{AB}$ is the diameter of a circle O and $\angle ACB$ is an inscribed angle subtended by $\overset{\frown}{AB}$, then $m(\angle ACB) = 90^0$

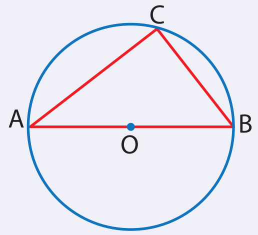
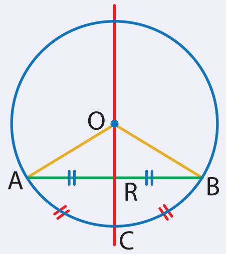



Figure 6.21

2 A line through the center of a circle perpendicular to a chord bisects both the chord and the arc intercepted by the chord.

In Figure 6.22, if O is the center of a circle and $\overline{OC}$ is perpendicular to chord $\overline{AB}$ at R, then $\overline{AR} \cong \overline{RB}$ and $m(\overset{\frown}{AC}) \cong m(\overset{\frown}{BC})$




3 A perpendicular bisector of any chord of a circle passes through the center of the circle.

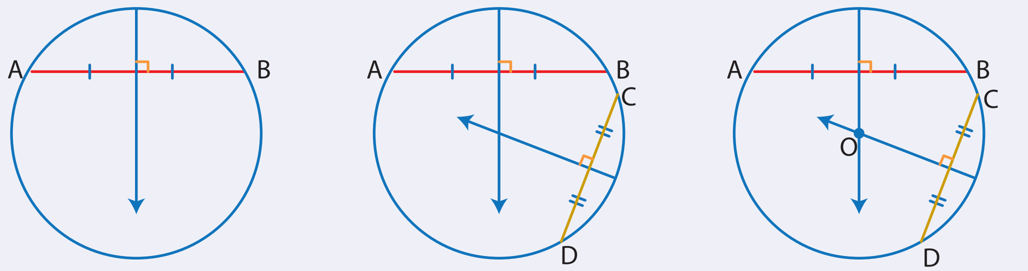
4 The center of a circle is located on the intersection point of perpendicular bisectors of the chords of the circle.

5 The center of any circle can be located by the following steps.

**Step 1:** Draw two different non parallel chords $\overline{AB}$ and $\overline{CD}$ of the circle.

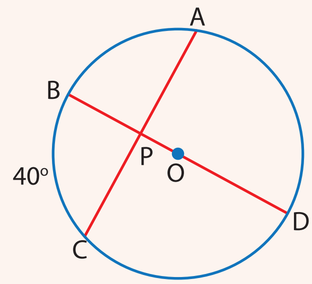
**Step 2:** Construct the perpendicular bisectors of these chords using compass and ruler.

**Step 3:** The intersection point of the perpendicular bisectors of $\overline{AB}$ and $\overline{CD}$ is the center of a circle.

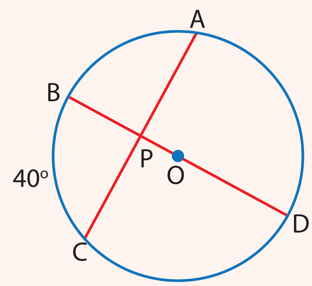


## Exercise 6.3

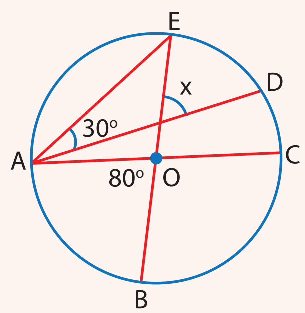
1 In the figure below, $\overline{AC}$ and $\overline{BD}$ are chords and O is the center of a circle. If $m(\widehat{AD}) = 72^0$ and $m(\widehat{BC}) = 40^0$, then determine m(∠APD)



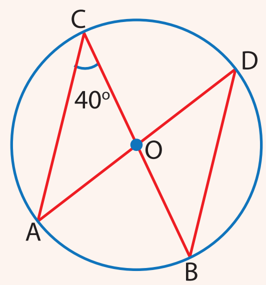

2 In the figure below, $\overline{AC}$ and $\overline{BD}$ are chords of the circle with center O. If $m(\widehat{ABC}) = 100^0$ and $m(\widehat{BC}) = 40^0$, then find m(∠APD)



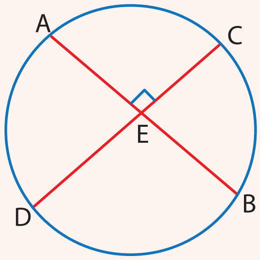

3 In the figure below, $\overline{AC}$, $\overline{AD}$, $\overline{AE}$ and $\overline{BE}$ are chords and O is the center of the circle. If $m(\angle AOB) = 80^0$ and $m(\angle EAD) = 30^0$, then find the value of $x$.



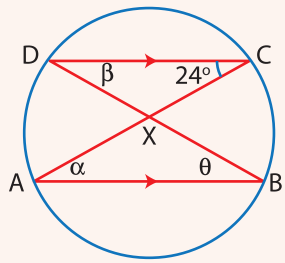

4 In the figure below, $\overline{AC}$, $\overline{AD}$, $\overline{BC}$ and $\overline{BD}$ are chords and O is the center of the circle. Find m(∠AOC) and m(∠CBD).



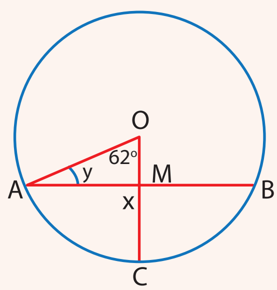
5 In the figure below, $\overline{AB}$ and $\overline{CD}$ are chords of the circle. If m(∠AEC) = $90^0$ and m(∠BAC) = $35^0$, then find m(∠ABD).




6 Find the values of all the unknown angles α,β and θ in the figure below if $\overline{AC}$ and $\overline{BD}$ are chords of the circle intersecting at X, m($\angle AXB$) = $100^0$ and m(∠ACD) = $24^0$




7 In the figure below, if Q is the center of the circle, $\overline{AB}$ is a chord of the circle, m($\angle AQC$) = $62^0$ and M is the midpoint of chord AB, then find the values of $x$ and $y$.

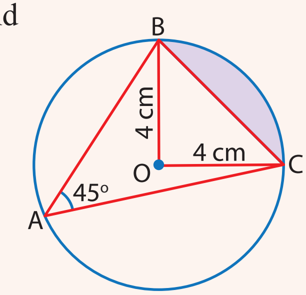



8 In the figure below, m($\angle BAC$) = $45^0$, O is the center of the circle with radius 4cm, $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$ are chords of the circle. Find

a m($\angle BOC$)

b m($\widehat{BAC}$)

c the length of chord $\overline{BC}$



# Unit Summary

1. A secant line intersects a circle at two distinct points.
2. A tangent line touches a circle only at a point.
3. The longest chord of a circle is called the diameter.
4. The measure of a minor arc of a circle is less than $180^0$.
5. The measure of a major arc of a circle is greater than $180^0$.
6. The measure of a semi-circles is $180^0$.
7. Arcs of the same circle are proportional to their corresponding angles.
8. A sector is part of a circle enclosed by two radii of a circle and its intercepted arc.
9. A segment of a circle is the region bounded by a chord and the arc subtended by the chord.
10. The measure of a central angle is equal to the measure of the arc it intercepts.
11. The measure of an inscribed angle is equal to half of the measure of the arc it intercepts.
12. Inscribed angles subtended by the same arc are congruent.
13. The measure of central angle subtended by a semi-circle is $180^0$.
14. The measure of an inscribed angle subtended by a semi-circle is $90^0$.
15. Two chords intersecting inside a circle form an angle whose measure is equal to half of the measures of opposite arcs subtending the angle.
16. A perpendicular bisector of any chord of a circle passes through the center of the circle.

## Review Exercises

1 Identify which of the following statements are true

a Every chord is a diameter of the circle.

b A bisector of any chord of a circle is the bisector of its arc.

c An angle inscribed in a semicircle is a right angle.

2 Two chords intersect inside a circle andform an angle of $75^0$. If one of the intercepted arc is $60^0$, find the measure of the other intercepted arc.

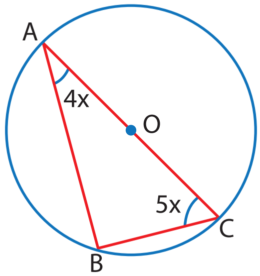
3 A chord of length 16cm is at a distance of 4cm from the center of the circle. Find the radius of the circle.

4 Show that a line through the center of a circle and perpendicular to a chord

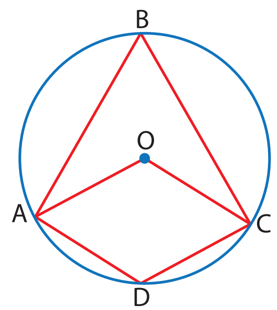
a bisects the chord.

b bisects the arc whose end points are the end points of the chord.

5 In the figure below, O is the center of the circle. Find the value of $x$.




6 In the figure to the right, O is the center of the circle




a If m (∠AOC) =$140^0$, then find m(∠ABC) and m(∠ADC)

b If m(∠ABC) = $60^0$, then find m(∠AOC) and m(∠ADC)

c If m(∠ABC) = $80^0$, then find m(∠OAC).

d If m(∠OCA) = $20^0$, then find m( $\overparen{ADC}$ )

7 In the figure below, if m(∠BFC) = 58°, m(∠BXG) = 48° and m(∠CBF) = 22°, then find the measure of

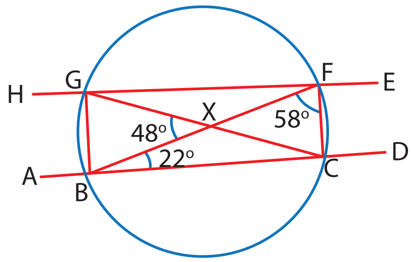
a ∠BGX

b ∠BGF

c ∠BCF

d ∠BCG

e ∠BFG




8 In the figure below if $\overline{AC}$ is diameter of the circle, m(∠CAD) = $25^0$ and m(∠BCD) = $132^0$, then find

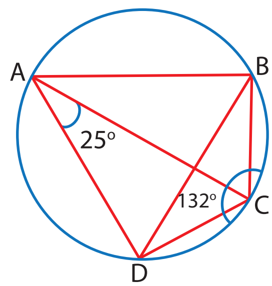
a m(∠ACD)

b m(∠ACB)

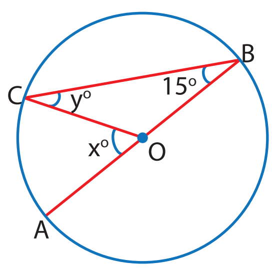
c m(∠BDC)

d m(∠CBD)

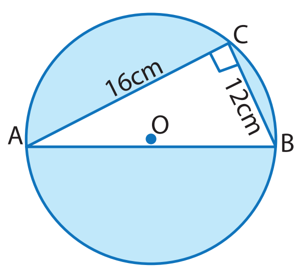
e m(∠CAB)




9 In the figure below, O is the center of the circle. If m(∠CBA) = $15^0$, then find the values of x and y.

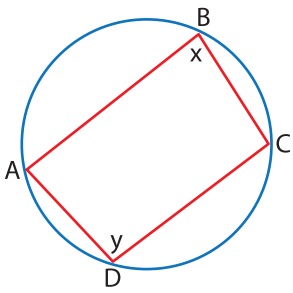



10 The diagram below shows a triangle ABC inscribed in a circle O of diameter AB, AC = 16cm and BC = 12cm. Find the area of the shaded region.



11 The tires of a car rotate 1000 revolution to cover a certain distance. If the diameter of the tire is 100 cm. how much distance did the car travel to cover that distance?

12 In the figure below, ABCD is an inscribed quadrilateral, express x in terms of y and y in terms of x.



# Unit 7

# SOLID FIGURES AND MEASUREMENTS

## Learning outcomes:

**After completing this unit, you will be able to:**

- identify types and parts of solid figures
- find the surface area of solid figures
- find the volume of solid figures
- appreciate application of solid figures in solving real-life problems.

## *Key terms*

- prism
- cylinder
- net of solid figures
- volume
- pyramid
- cone
- surface area
- slant height

## Introduction

In your previous grades, you have studied about perimeters and areas of squares, rectangles, triangles, circles, sectors of circles, etc. These figures are called plane figures because each of them lies in a plane. However, most of the objects that you come across in daily life do not wholly lie in a plane. Some of these objects are match box, balls, ice cream cones, drums, and so on. The figures representing these objects are called **three dimensional** **es** or **solid** **es**. Some common solid figures are prisms, pyramids, cylinders, cones and spheres. In this unit, you will learn about the shape, surface areas and volumes of common solid figures and their applications.

## 7.1 Solid Figures

A solid figure is a three-dimensional figure because it has three dimensions length, width and height. Solid figures can be identified by the shape of their bases, number of faces and the shape of their lateral faces. In this section, you will learn about the most common solid figures namely prism, pyramid, cylinder and cone.

### Activity 7.1

1 Give a model or real object as an example for each of the solid figures in Figures 7.1.

2 For each of the solid figures identify the type and number of faces.

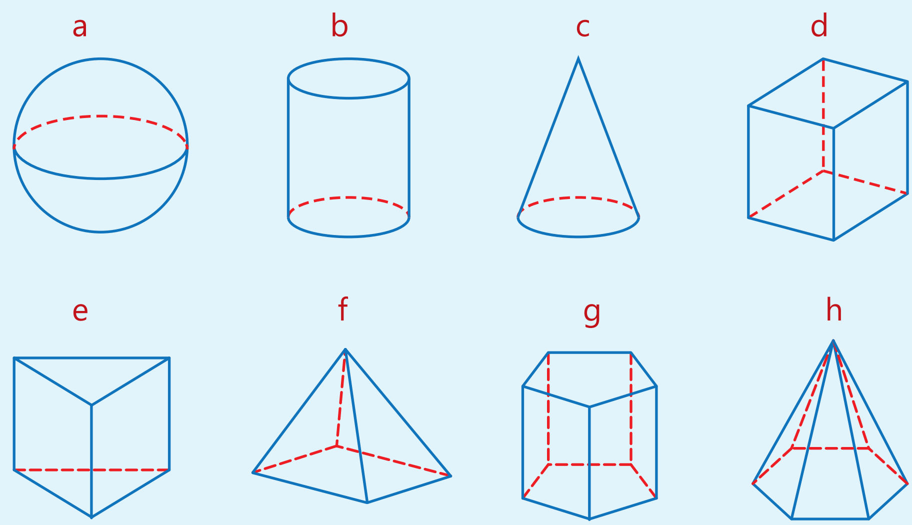



Figure 7.1 Solid figures

You can mention real objects having shapes similar to the above figures in your surroundings. For example ball, tin can, match box, funnel, roof of hut, barrel, drum and so on. A match box has six rectangular faces but a tin can has curved face.

### 7.1.1 Prism and Cylinder

## Activity 7.2

A designer wants to make containers in the shape of a box and a tin can as shown in Figure 7.2.

a How many bases do the box and the tin can have?

b What are the shapes of the bases for the box and the tin can?

c What are the shapes of the lateral faces of the box and the tin can?

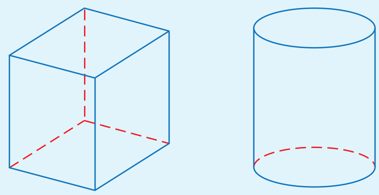
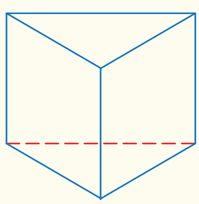


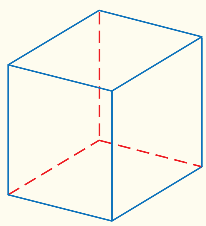
Figure 7.2

### Definition 7.1

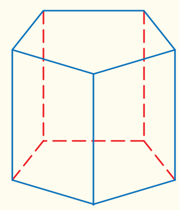
A prism is a solid figure with two congruent parallel faces called bases, that can be triangles, squares, rectangles or a polygon in general and the lateral faces are parallelograms. A prism take its name from the name of the base.



Triangular Prism
(Bases are Triangles)



Square Prism
(Bases are Squares)



Pentagon Prism
(Bases are Pentagons)

A prism has two congruent bases: upper base and lower base. The lateral faces of any prism are parallelograms. The edges of a prism are line segments that are intersection of adjacent faces. The intersection part of edges of the prism is called vertex of the prism. Consider the rectangular prism shown in Figure 7.3.

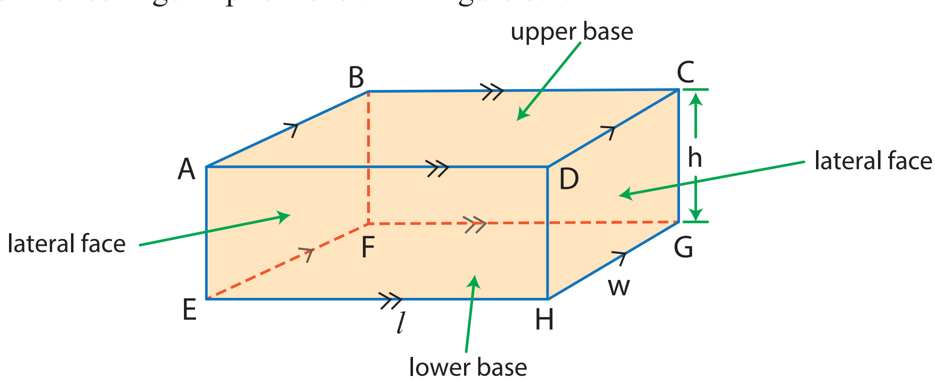



Figure 7.3

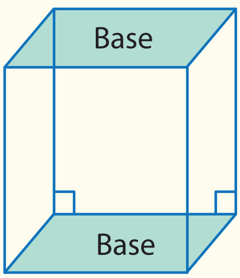
In Figure 7.3

- The quadrilateral region ABCD is the upper base and quadrilateral EFGH is the lower base. The two quadrilateral are parallel and congruent.

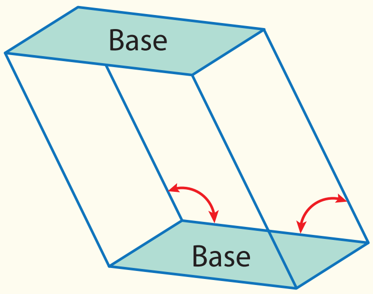
- The parallelograms ABFE, BCGF, CDHG and ADHE are lateral faces of the prism.
- The line segments $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}, \overline{EF}, \overline{FG}, \overline{GH}$ and $\overline{HE}$ are edges of the base. and the line segment $\overline{AE}, \overline{BF}, \overline{CG}$ and $\overline{DH}$ are lateral edges of the prism.
- Points A, B, C, D, E, F,G and H are verices of prism.

## Definition 7.2

A prism is said to be right prism, if each lateral edge is perpendicular to the edge of the base. Otherwise it is called an Oblique prism.




Right Prism



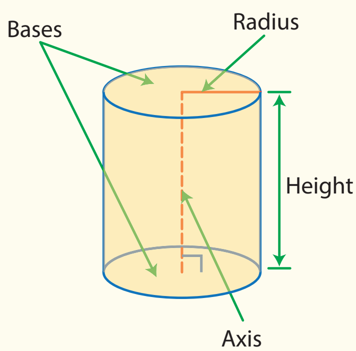

Oblique Prism

A right prism has the following properties.

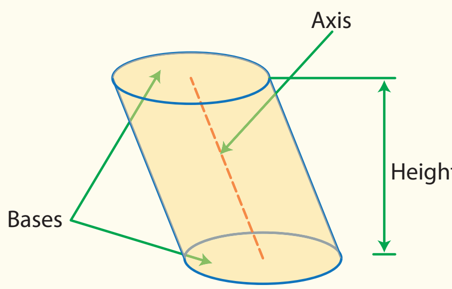
- The bases are congruent polygons.
- The bases lay on parallel planes.
- Every lateral edge is perpendicular to its base edge.
- All lateral edges are parallel and congruent.
- All lateral faces are rectangular regions.

## Definition 7.3

A cylinder is a solid figure whose upper and lower bases are congruent circles joined by curved surface on parallel planes.




Right cylinder



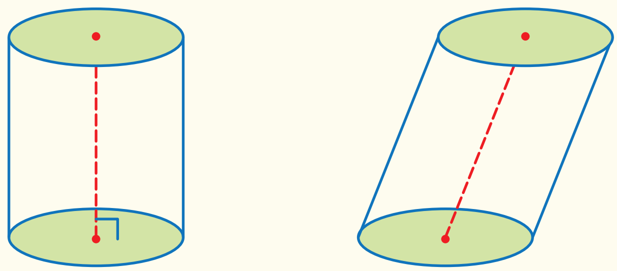

Oblique cylinder

## Note

The distance between the two bases of the cylinder is called height of the cylinder.

### Definition 7.4

A right cylinder is a cylinder in which the line through the center of the bases is perpendicular to the bases, otherwise it is called an oblique cylinder.



Right Cylinder Oblique cylinder

A right circular cylinder has the following properties.

- The upper and the lower bases are congruent (circles of equal radius).
- The bases lay on parallel planes.
- A line through the centers of the bases is perpendicular to every diameter of the bases.

## Exercise 7.1

1 Answer the following questions based on the given information in Figure 7.4

a Write the name of the solid figure

b Identify all bases, lateral faces, lateral edges and vertices

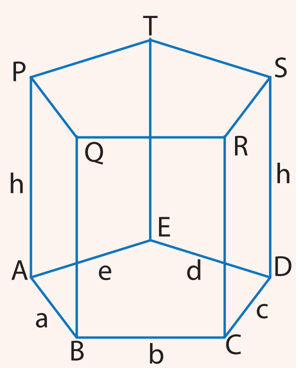



Figure 7.4

2 Draw a right hexagonal prism and identify its parts

3 Draw a right cylinder and identify its parts

### 7.1.2 Pyramids and Cones

## Activity 7.3

1 The roof of a house is designed in two different models as shown in Figure 7.5 below. What similarity and difference do you observe?

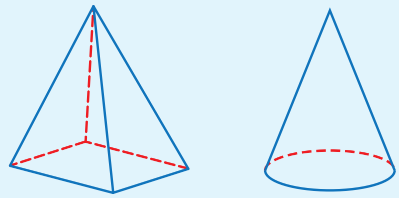


Figure 7.5

2 Have you ever seen the shape in Figure 7.6? What is its name?

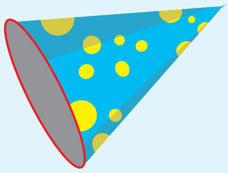
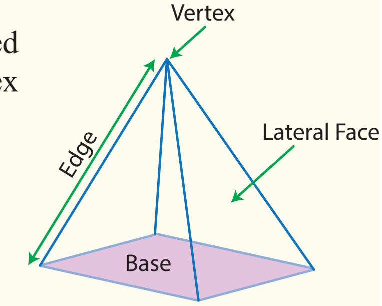


Figure 7.6

### Definition 7.5

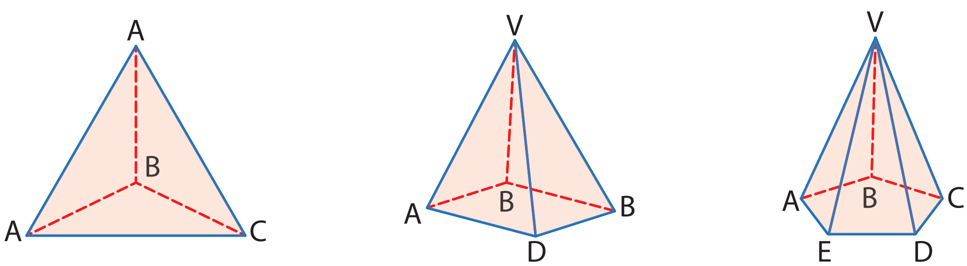
A pyramid is a solid figure bounded by a polygon called the base, and others are triangles having a common vertex at some point not in the plane of the base.




pyramid

The base of a pyramid can be triangle, quadrilateral or any polygon and the corresponding pyramid is named according to its base.

- Triangular pyramid if its base in a triangle.
- Rectangular pyramid if its base is a rectangle.
- Square pyramid if its base is a square.




Triangular Pyramid Rectangular Pyramid Pentagon Pyramid

In this discussion we shall consider only pyramids with regular bases. A regular pyramid is a pyramid whose base is a regular polygon and whose lateral edges are all equal in length and as a result all the lateral faces are congruent isosceles triangles.

Consider the pyramid Figure 7.7

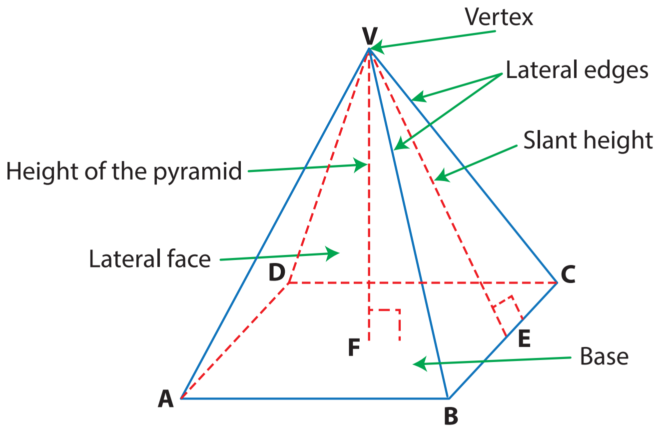
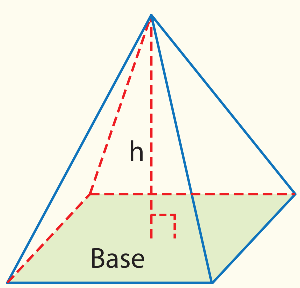



Figure 7.7 Regular pyramid

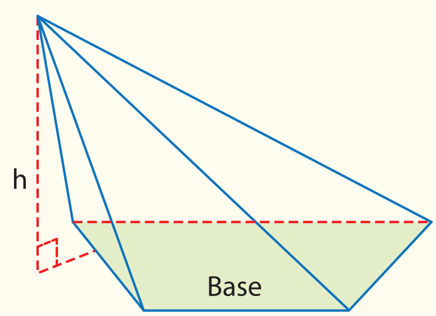
- The polygonal region ABCD is the base of the pyramid.
- The point V outside of the plane of the polygon (base),which is the intersection of the lateral faces, is the vertex of the pyramid.
- The triangles ΔVAB, ΔVBC, ΔVCD and ΔVDA are lateral face of the pyramid.
- $\overline{AB}, \overline{BC}, \overline{CD}$ and $\overline{DA}$ are the edges of the base of the pyramid
- $\overline{VA}, \overline{VB}, \overline{VC}$ and $\overline{VD}$ are lateral edges of the pyramid, they meet at the vertex
- The perpendicular distance VF from the vertex to a point on the base is the altitude of the pyramid.
- The length of the altitude of a lateral face of the pyramid is a slant height.

## Definition 7.6

A right pyramid is a pyramid whose altitude drawn from its vertex to the base passes through the center of the base and it is perpendicular to the base, otherwise it is called oblique pyramid.




Right Pyramid




Oblique pyramid

A right pyramid has the following properties.

- The foot of the altitude is at the center of the base.
- All the lateral faces are isosceles triangles.
- All the slant heights are equal.

## Definition 7.7

Cone is a solid figure formed by a circular shaped base and closed curved surface meet at a point not on the plane of the base.

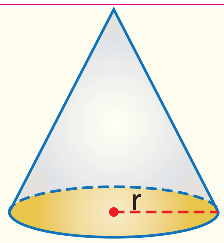



Figure 7.8 A cone

Consider the cone in Figure 7.9.

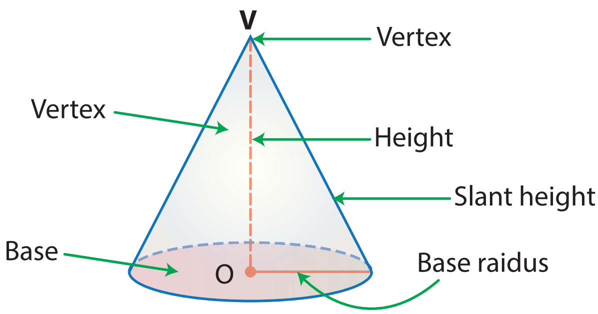
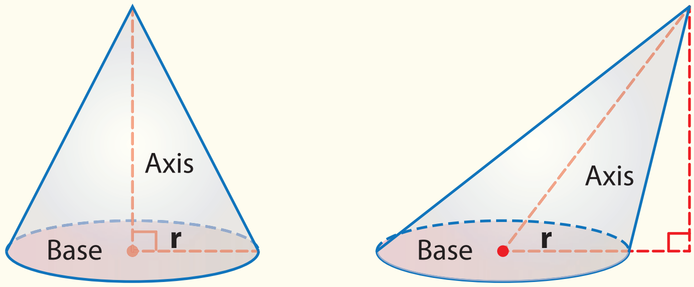



Figure 7.9

- The circle O is the base of the cone.
- The closed curved surface is lateral surface of the cone.
- The point V outside the plane of the base at which the curved surface meets is the vertex of the cone.
- The perpendicular distance VO from the vertex to the base is called the altitude of the cone.
- The length of a line segment from the vertex to any point of the base circle is slant height.

## Definition 7.8

A right circular cone is a circular cone with the foot of its altitude is at the center of the base, otherwise it is called oblique cone.




Right Cone    Oblique Cone

A right cone has the following properties.

- The altitude is perpendicular to the base at the center of the circle.
- Any two slant heights are congruent.

## Exercise 7.2

1 Answer the following questions based the information in Figure 7.10

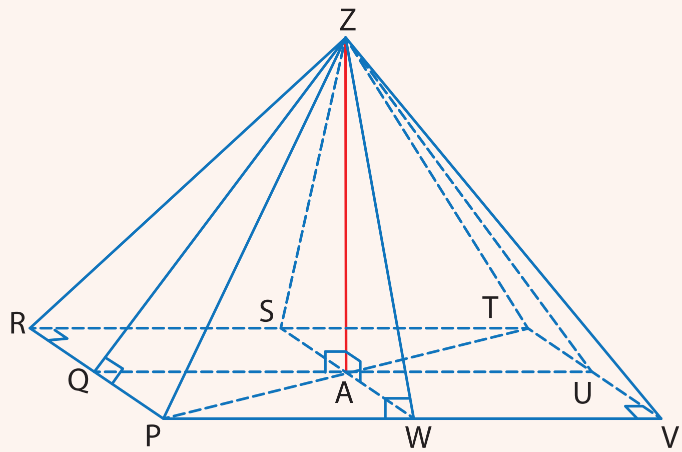



Figure 7.10

a Give a name for the solid figure.

b Identify the base, lateral faces, lateral edges , vertex, slant height and height of the pyramid.

2 Draw a right pentagonal pyramid and identify its parts

3 Draw a right circular cone and identify its parts.

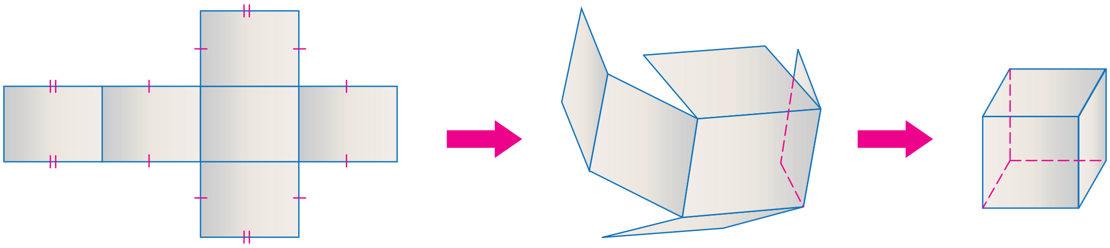
# 7.2 Surface Area of Solid Figures

In our daily life, there are many situations which need the concept of surface area of solid figures. For example, if you are interested in paint the walls and ceiling of a room, you have to find the surface areas of the walls and the ceiling of the room to determine the cost.

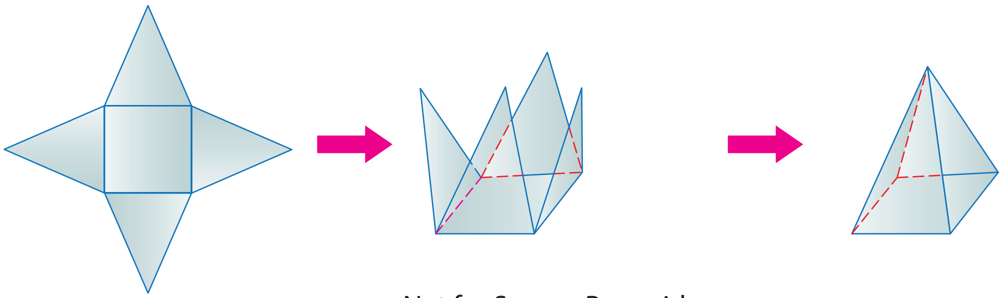
### 7.2.1 Nets of Solid Figures

A net is a two-dimensional figure that can be folded into a solid figure. There are many different nets. Here, nets have been provided for the cube, square pyramid, rectangular prism, triangular prism, triangular pyramid, cylinder and cone. Cutting out these nets, folding and gluing them to create a solid object, will help you become familiar with the features of these solids (such as their faces, edges and vertices). A solid can have several different nets.

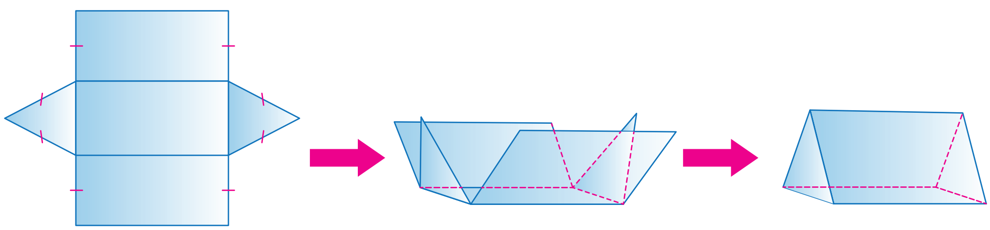
## Example 7.1



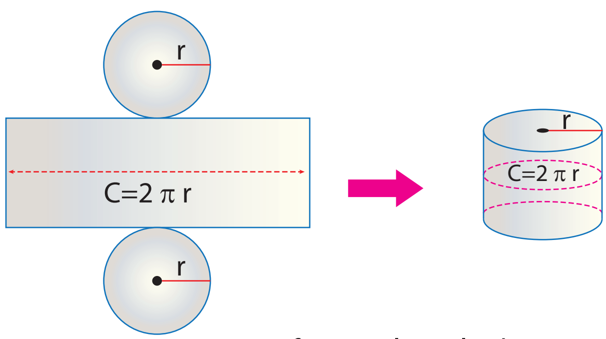
Net for Square Prism



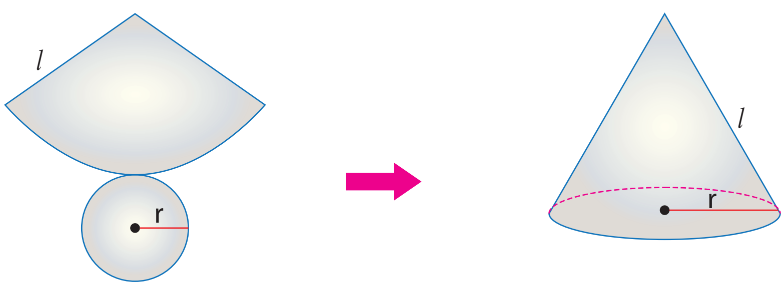
Net for Square Pyramid



Net for Triangular Prism



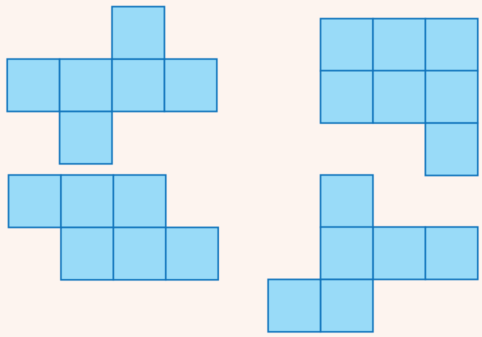

Net for Circular Cylinder



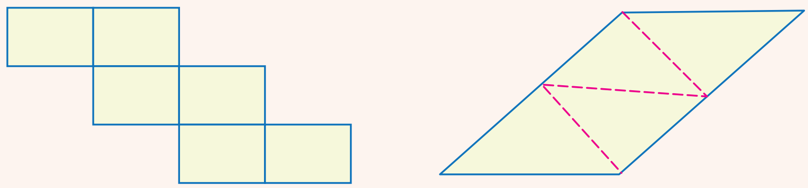

Net for Circular Cone

## Exercise 7.3

1 Which of the following are nets of a cube?



2 Prepare papers with the following design and fold to identify solid figures that could be formed by each of the following nets.



3 Use the description below; identify an object that has each set of faces.

a Two congruent squares and four congruent rectangles.

b One rectangle and two pairs of congruent triangles.

c Four congruent equilateral triangles.

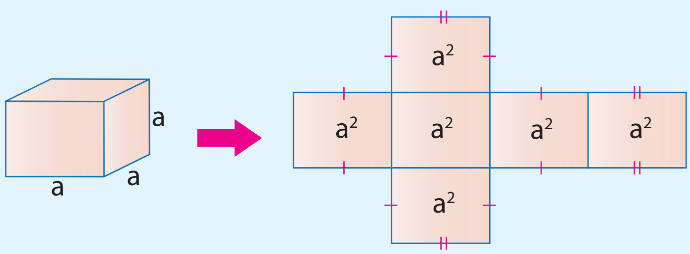
### 7.2.2 Surface Area of Prism and Cylinder

## Activity 7.4

The figure below shows a cube with edge length a units and its nets (all faces) are given.

a Calculate the area of each face of the cube and add together. Compare this result with the area of the net of the cube?

b How much faces are needed to make net of a cube?




From your response in Activity 7.4 observe that the sum of the areas of all faces of the cube is equal to the area of the net of the cube.

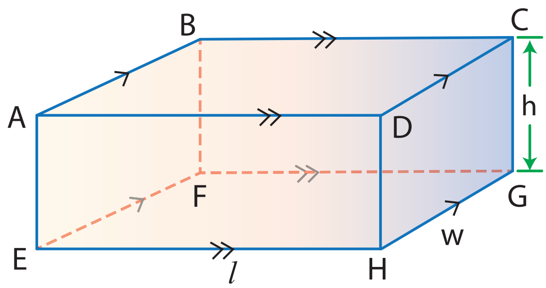
**Definition 7.9**

The surface area of a solid figure is the sum of the area of each face of the solid figure.

## Surface Area of a Rectangular Prism

You can use a net to help you to find the surface area of the solid figure.

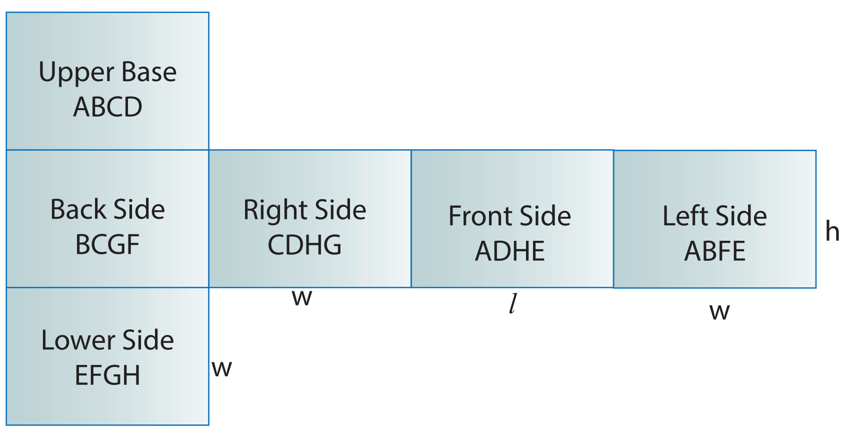
Consider a rectangular prism with length $\ell$, width w and height h.




Then, the lateral faces of the prism are rectangles ABFE, BCGF, CDHG and ADHE and the bases are rectangles ABCD and EFGH.

Now, you can compute surface area using nets as follows.

**Step 1:** Draw the net. A rectangular prism has six faces.That is, two bases (Lower and upper) and four lateral faces.




**Step 2:** Find the area of each face

Face ABCD: $Area(ABCD)=\ell w$ Face BCGF: $Area(BCGF)=\ell h$

Face EFGH: $Area(EFGH)=\ell w$ Face CDHG: $Area(CDHG)=wh$

Face ADHE: $Area(ADHE)=\ell h$ Face ABFE: $Area(ABFE)=wh$

**Step 3:** Add all the areas to find the surface area. Let $A_L$ be the lateral surface area,

Then

$$A_L = a(ADHE) + a(BCGF) + a(CDHG) + a(ABFE)$$

$$A_L = \ell h + \ell h + wh + wh$$

$$A_L = 2(\ell + w)h$$

$$A_L = Ph, \text{ where } P \text{ is perimeter of the base}$$

Let $A_B$ be area of one of the bases. Then $A_B = \ell w$

Let $A_T$ be the total surface area of the regular rectangular prism. Then

$$A_T = A_L + 2A_B$$

$$A_T = Ph + 2A_B = 2(\ell + w)h + 2\ell w$$

$$= 2(\ell h + wh + \ell w)$$

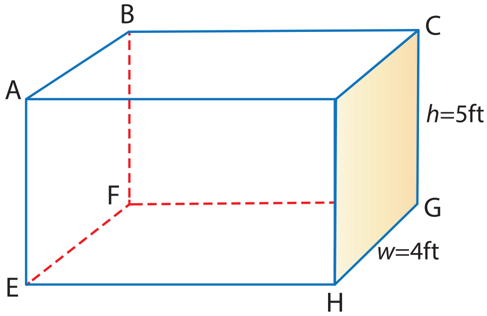
Therefore, the total surface a rectangular regular prism with length $\ell$, width w and length h is given by $A_T = 2(\ell w + \ell h + wh)$

## Example 7.2

A carpenter is designing a wooden box for sale. The box has length 6ft, width 4ft and height 4ft. Before he buys the wood, he needs to find the total surface area of the box. How much square feet of material is needed to prepare the box?

### Solution

Suppose the box is designed as in the following figure.




Then it is given that $\ell = 6\,\text{ft}$, $w = 4\,\text{ft}$ and $h = 5\,\text{ft}$.

Then the total surface area of the box is

$A_T = 2(\ell w + \ell h + wh) = 2(6\text{ft} \times 4\text{ft} + 6\text{ft} \times 5\text{ft} + 4\text{ft} \times 5\text{ft})$

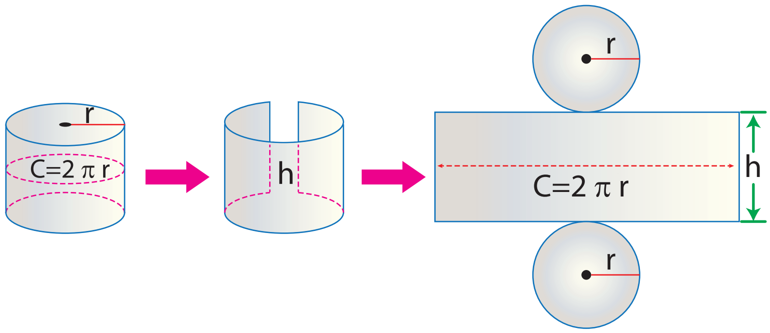
$= 2\ (24\text{ft}^2 + 30\text{ft}^2 + 20\text{ft}^2)$

So, $A_T = 2(74\text{ft}^2) = 148\text{ft}^2$

Therefore, the carpenter has to buy a wooden material of 148 $\text{ft}^2$ to prepare the box of his design.

## Surface Area of a circular cylinder

Consider a circular cylinder with base radius r and height h. The net for this solid is shown in the figure below.




Since the bases of a cylinder are congruent circles, they have equal area. If the bases are detached, the remaining parts of the cylinder form a rectangle of length 2πr (Circumference of the circle) and width h (height of the cylinder)

Thus, we have;

the lateral surface area of the cylinder is the area of the rectangle:

$A_L = 2\pi rh$

the area of the bases is the sum of the areas of the two circles:

$2A_B = 2\pi r^2$

The total surface area of the cylinder is

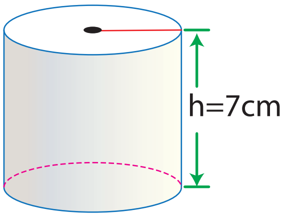
$A_T = A_L + 2A_B = 2\pi rh + 2\pi r^2 = 2\pi r\,(r+h)$

## Example 7.3

If the height of a right circular cylinder is 7 cm and diameter of its base is 10cm, then find

a lateral surface area of the cylinder

b total surface area of the cylinder,




### Solution

Since diameter of the base $d$=10cm, its radius $r$ =5cm and height $h$=7cm. Then

a Lateral surface area:

$A_L = 2\pi rh = 2\pi(5\text{cm})\ (7\text{cm})$

$A_L = 2\times\pi\times 35\text{cm}^2 = 70\pi\text{cm}^2$

Therefore, lateral surface area of the cylinder is $70\pi\text{cm}^2$

b Total surface area:

$A_T = 2\pi r(r+h)$

$A_T = 2\pi(5cm)(5cm+7cm)$

$A_T = 2 \times \pi \times (5cm)(12cm)$

$A_T = 120\pi cm^2$

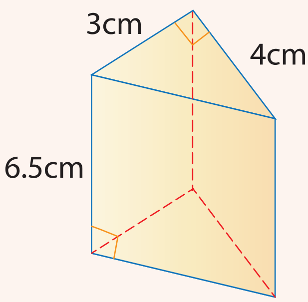
Therefore, total surface area of the cylinder is $120\pi cm^2$

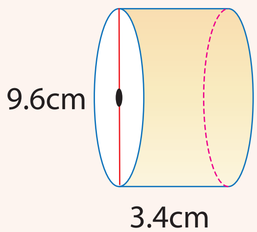
## Exercise 7.4

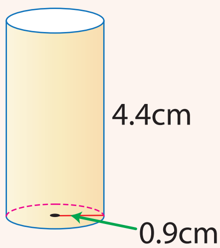
1 Find the lateral and total surface area of a rectangular prism whose height is 8cm and base with length 4cm and width 6cm.

2 Find the lateral surface area and total surface of a right circular cylinder of radius 5m and height 1.4 m.

3 Find the total surface area for the following solid figures




### 7.2.3 Surface Area of Pyramid and Cone

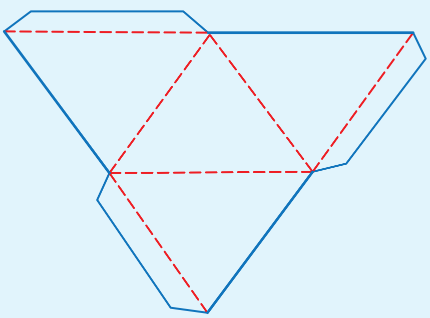
## Activity 7.5

Prepare a net of triangular pyramid by using piece of paper as shown below.

a Construct the model of the pyramid by using the above net.

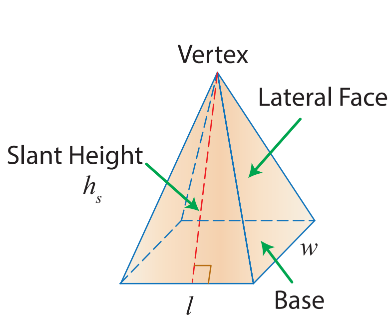
b What is the shape of each face of the pyramid?

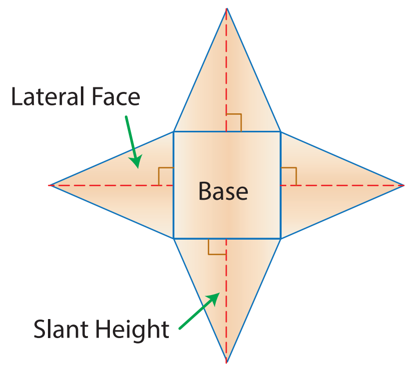
c Discuss how to find the surface area of pyramid?



The lateral surface area of pyramid is the sum areas of its lateral faces. The total surface area of a pyramid is the sum of its lateral surface area and the area of is base

Consider the net diagram of a square pyramid as shown below




The lateral surface area $A_\ell$ of a square pyramid is given by= $4\left(\frac{1}{2}h_s\ell\right)$ (the four lateral faces are congruent triangles with $\ell$ as base and $h_s$ as a height ) and the total surface area $A_T$ of a square pyramid is given by

$$A_T = A_B + A_\ell = l^2 + 4\left(\frac{1}{2}h_s\ell\right) = l^2 + \left(\frac{1}{2}Ph_s\right)$$

where,

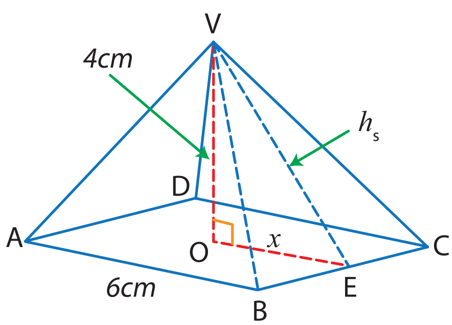
- $\ell$ is the length of one side of the square base;
- $h_s$ is the slant height of the pyramid; and
- P is perimeter of the base, P=4$\ell$

## Example 7.4

The base of right square pyramid has length 6cm. If the height of the pyramid is 4cm, then find

a lateral surface area of the pyramid;

b total surface area of the pyramid.




**Solution**

Let us draw the model of the square pyramid with given dimensions.

To find the lateral surface area, we need the length of the slant height. Since ∠VOE is right angle triangle, by Pythagoras theorem

$$(VE)^2=(VO)^2+(OE)^2=(VO)^2+\left(\frac{AB}{2}\right)^2 \text{ (VE is the slant height and } OE=\frac{1}{2}AB\text{)}$$

$$(VE)^2=(4\text{cm})^2+(3\text{cm})^2=25\text{cm}^2$$

Thus, $h_s = VE=\sqrt{25\text{cm}^2}=5\text{cm}$.

Hence, the lateral surface area is given by

$$A_L=\frac{1}{2}Ph_s=\frac{1}{2}\times(4\times6\text{cm})\times5\text{cm}$$

$$A_L=60\text{cm}^2$$

The base is square with side 6cm.

So, the base area is $A_B = (6\text{cm})^2$

$=36\text{cm}^2$

Thus, the total surface area

$$A_T = A_L + A_B=60\text{cm}^2+36\text{cm}^2=96\text{cm}^2$$

## Surface Area of a Cone

Consider a right circular cone with base radius $r$ and slant height $h_s$. Then its net is given in Figure 7.11. The net of a cone is union of a circle and a sector.

**Note:** the area of sector $A_s$ of a circle of radius is r and arc length $\ell$ is given by the formula $A_S=\frac{1}{2}r\ell$

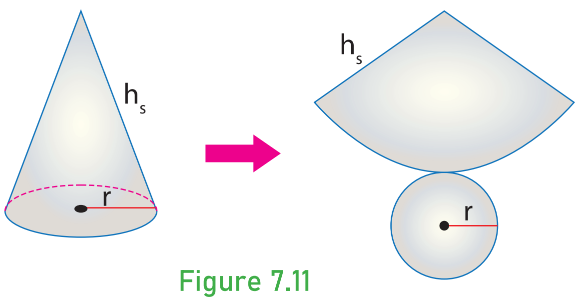



Figure 7.11

The base area is the area of a circle of radius r. That is $A_B=\pi r^2$

The lateral surface area is the area of the sector which is given by

$A_L=\frac{1}{2}\times h_s\times2\pi r=\pi rh_s$, where r is given by the radius of the sector and $2\pi r$ is the length of the arc.

Hence, the total surface area

$$A_T = A_L + A_B = \pi rh_s + \pi r^2 = \pi r(r+h_s)$$

**Note:** The area of a sector of a circle with radius r and central angle $\theta$ is defined by

$$A = \frac{\pi r^2 \theta}{360^0}$$

## Example 7.5

Find the total surface area of a right circular cone whose altitude is 12cm and base radius is 5cm.

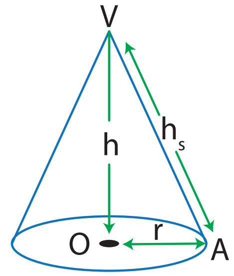


Figure 7.12

### Solution

Consider Figure 7.12

Since the altitude is h=12cm and the base radius is r=5cm, then using Pythagoras theorem on the right angled triangle ΔVOA, the slant height $h_s$, can be calculated as

$$(VA)^2 = (OA)^2 + (VO)^2$$

$$h_s^2 = r^2 + h^2 = (5\text{cm})^2 + (12\text{cm})^2 = 25\text{cm}^2 + 144\text{cm}^2 = 169\text{cm}^2$$

Thus, $h_s = \sqrt{169\text{cm}^2} = 13\text{cm}$

Then, the total surface area is given by

$$A_T = \pi r(r + h_s) = \pi \times 5cm \times (5cm + 13cm)$$

$$A_T = 5\pi\, cm \times 18cm = 90\pi\, cm^2$$

## Exercise 7.5

1 Find lateral surface area and total surface area of a right square pyramid with length of the base 8cm and slant height of the pyramid is 10 cm.

2 Find lateral surface area and total surface area of a right circular cone with diameter of the base 12 cm and height 8cm.

3 If the edge of the base of a regular square pyramid is 16cm long and its lateral surface area is 320 $cm^2$, then

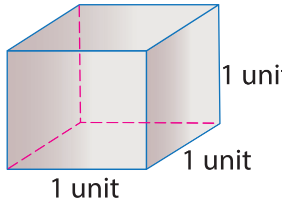
a find the slant height of the pyramid;

b the altitude of the pyramid.

4 What is the increase in the total surface area of a cone, if the base radius and slant height are both doubled?

## 7.3 Volume of Solid Figures

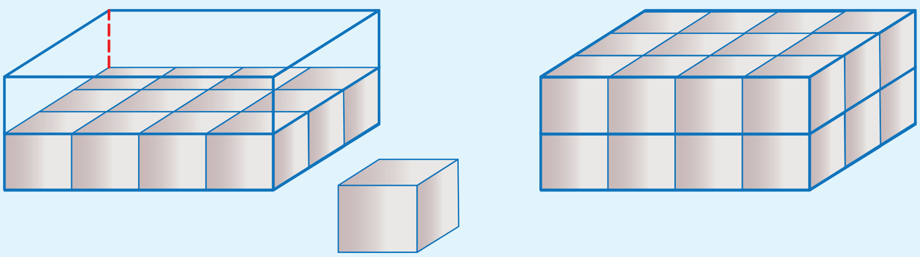
Volume is the quantity of three-dimensional space occupied by a liquid, solid, or gas. Common units used to express volume include liters, cubic meters ($m^3$), gallons, cubic centimeters ($cm^3$). To find the volume of three-dimensional figure, we can use cubes as a unit and count the number of cubes it should take to fill the space.




### 7.3.1 Volume of Prism

**Activity 7.6**

Consider the following rectangular prism filled with unit cubes.



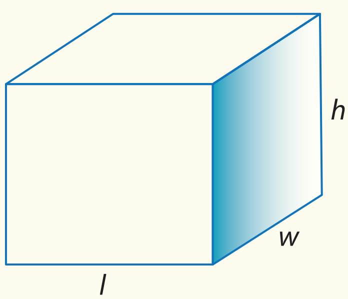
a How many unit cubes are needed to fill the object in the left?

b Set a formula to determine the total number of unit cubes contained in the prism?

From your responses in Activity 7.6, observe that the the total numberof cubes that are used to fill the rectangle box is the volume of the box.

**Definition 7.10**

The volume (V) of a right prism is the product of its base area $A_B$ and height (h) of the prism. That is, $V= A_B h$




The volume V of a right rectangular prism of length l, width w and height h is given by V=lwh

## Example 7.6

Find the volume of a rectangular prism of length 3cm, width 4cm and height 5cm

### Solution

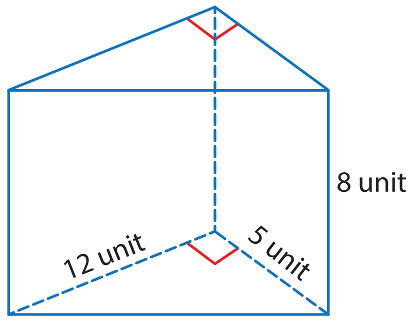
Volume = $l \times w \times h$

$= 3\text{cm} \times 4\text{cm} \times 5\text{cm}$

$= 60\text{cm}^3$

## Example 7.7

Find the volume of a right prism whose altitude is 8 units and whose base is a right-angled triangle with legs 5 units and 12 units.




### Solution

The prism is illustrated in the Figure in the right side

$$V = A_B h$$

$$V = \left(\frac{1}{2} \times 5\text{cm} \times 12\text{cm}\right) 8\text{cm}$$

$$V = 30\text{cm}^2 \times 8\text{cm} = 240\text{cm}^3$$

## Example 7.8

If the volume of a square prism is B cm$^3$, then find the height of the prism in terms of dimension of the base and its volume.

### Solution

Let h be the height of a prism. If the length of side of the base is a cm, then

$$V = B\text{cm}^3 = A_B h$$

$$h = \frac{V}{A_B} = \frac{B}{A_B}\text{cm}^3, \text{but } A_B = a^2\text{cm}^2 \text{ (the base is square)}$$

$$\text{Then, } h = \frac{B\text{cm}^3}{a^2\text{cm}^2} = \frac{B}{a^2}\text{cm}$$

## Exercise 7.6

1. Find the volume of a square prism with edges of the square are 6cm each and its height is 10cm.
2. Find the length of the edge of a cube whose volume is $3375cm^3$
3. The length, width and height of a rectangular box are 12m, 8m and 6m respectively. How many small boxes can it hold if each box occupies 1.5 $m^3$ space?

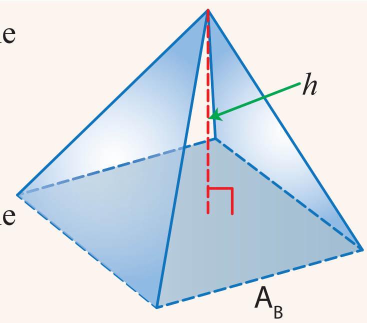
### 7.3.2 Volume of Pyramid

**Theorem 7.1**

The volume V of any pyramid is equal to one-third of the product of its base area and height. That is

$$V=\frac{1}{3}A_Bh$$

where, $A_B$ is the base area of the pyramid and $h$ is the height of the pyramid.




## Example 7.9

The base of a pyramid is a square whose side is 6cm long. Find the volume of the pyramid if its height is 4cm.

**Solution**

Since the base is a square, the base area is given by

$$A_B=(6\text{cm})^2=36\text{cm}^2$$

Then, volume of the pyramid is given by

$$V=\frac{1}{3}A_Bh=\frac{1}{3}\times 36\text{cm}^2\times 4cm=48\text{cm}^3$$

Thus, the volume of the pyramid is $48cm^3$

## Example 7.10

Find the height of a rectangular pyramid with volume of $100cm^3$ and a base of length 4cm and width 5cm.

**Solution**

Volume = $A_Bh$, where $A_B$ = area of the base and h= height

$A_B$= lw, where l is the length and w is the width of the base.

So, $A_B = 4cm \times 5cm = 20\ cm^2$

$100\ cm^2 = 20cm \times h$ and $h = \frac{100\,cm^2}{20\,cm}$

$= 5\,cm$

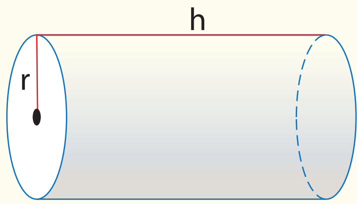
## Exercise 7.7

1. Find the volume of a right pyramid whose altitude is 6cm and whose base is a right angled triangle with legs 3cm and 4 cm.
2. Find the length of edge of the base of square pyramid whose volume and height are $1200cm^3$ and 25cm respectively.

### 7.3.3 Volume of Cylinder

**Definition 7.11**

The volume V of a circular cylinder is equal to the product of the base area and its altitude. That is $V = A_B h = \pi r^2 h$ where, r is the radius of the base, h is the height of the cylinder




## Example 7.11

A tin can has a shape of circular cylinder whose base radius is 5cm and height is 80cm. Find the volume of the tin can.

**Solution**

We know that a tin can is one of the real objects modeled by circular cylinder. Then its volume is given by

$$V = A_B h = \pi r^2 h$$

$$V = \pi \times (5cm)^2 \times 80cm = \pi \times 25cm^2 \times 80cm = 2000\pi cm^3$$

Thus, the volume of the tin can is $2000\pi cm^3$

## Exercise 7.8

1. A cylindrical water tank has a base diameter 7 m and height 2.1 m. Find the capacity (volume) of the tank.
2. Length and width of a rectangular paper are 33 cm and 16 cm respectively and it is folded about its width to form a cylinder. Find the volume of the cylinder.

### 7.3.4 Volume of Cone

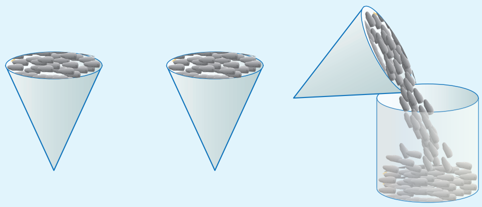
## Activity 7.7

Do the Activity in a group of two or three students.

Using hard paper, construct a right circular cylinder and a cone having the equal base and height. Then

a Fill the cone with sand or any cereals and pour into the cylinder. How many times did you pour into the cylinder to make it full?

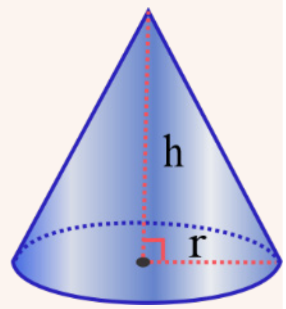
b What can you say about the volume of a cylinder and a cone?



From your responses in Activity 7.7, observe that the cylinder can be filled with three cones sand or cereal

**Theorem 7.2**

The volume V of a right circular cone with height h and radius of the base r is equal to one-third of the product of the base area and its altitude. That is, $V = \frac{1}{3}A_B h = \frac{1}{3}\pi r^2 h$ where, r is the radius of the base and h is the height of the cone




## Example 7.12

Find the volume of right circular cone with height 12cm and base radius 5cm.

**Solution**

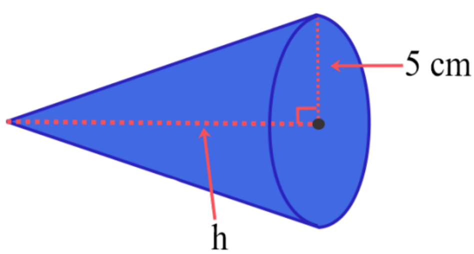
It is given that base radius $r$ =5cm and height $h$ =12cm

Hence, the volume of the cone is given by

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3} \times \pi \times (5\text{cm})^2 \times 12\text{cm}$$

$$V = 100\pi \ cm^3$$

Thus, the volume of the cone is $100\pi \text{ cm}^3$




## Exercise 7.9

1. Find the volume of right circular cone with slant height 13cm and base radius 5cm.
2. A cone has the same height and its radius is twice the radius of another cone. What is the ratio of the volumes of the two cones?

# Unit Summary

1 Family of prisms and pyramids have plane faces whereas family of cylinders and cones have curved and circular faces.

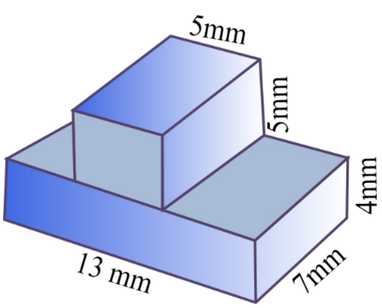
2 The surface areas and volume of solid figures (prism, pyramid, cylinder and cone) is summarized in the table below.

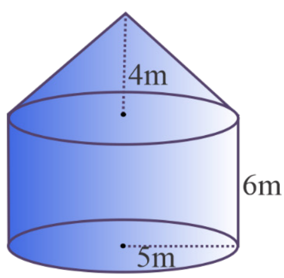
| Shape | Lateral surface area | Total surface area | Volume |
|---|---|---|---|
| Prism | $A_L = Ph$<br>$P$- perimeter of base<br>$h$-height of prism | $A_T = A_L + 2A_B$. | $V = A_B h$<br>$A_B$- area of base<br>$h$- height of prism |
| Pyramid | $A_L = \frac{1}{2} Ph_s$<br>$P$- perimeter of base<br>$h_s$- slant height of pyramid | $A_T = A_L + A_B$ | $V = \frac{1}{3} A_B h$<br>$A_B$- area of base<br>$h$- height of pyramid |
| Cylinder | $A_L = 2\pi rh$<br>$r$- radius of base<br>$h$- height of cylinder | $A_T = 2\pi r(r+h)$ | $V = \pi r^2 h$<br>$r$- radius of base<br>$h$- height of cylinder |
| Cone | $A_L = \pi rh_s$<br>r= radius of base<br>$h_s$-slant height of cone | $A_T = \pi r(r+h_s)$ | $V = \frac{1}{3}\pi r^2 h$<br>$r$- radius of base<br>$h$- height of cone |

## Review Exercises

1 How many plane faces are needed to form a solid figure?

2 Find the total surface area and volume of the following solid figures.



3 If the length of the edge of a cube is doubled, then find

a the increase in its surface area;

b the increase in its volume.

4 If the radius of the base of a cylinder r is doubled, then find

a the increase in its lateral surface area;

b the increase in its volume.

5 Two equally A4 sized sheets of papers are rolled one along the length and the other along the width to form cylinders. Which cylinder do you think has the greater volume? Explain.

6 Maritu wants to sell juice in cone-shaped cups. Each cup has a diameter of 4 cm and height of 8cm. She wants to make enough juice for 50 cups. How much juice does Martu need to make so all the cups are full?

7 A square sheet of paper 16cm by 16 cm is folded to form lateral surface area of a square prism. Find the area of the base.

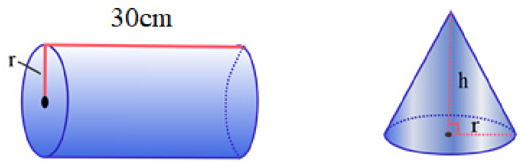
8 The volume of a rectangular prism is 1944cm$^3$. Find the dimensions if the edges are in the ratio of 3:4:6.

9 The volume of a right circular cylinder is 3080 cm$^3$ and radius of its base is 7cm. Then find

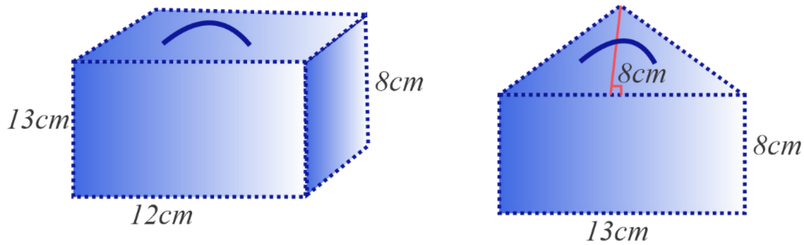
a the lateral surface area of the cylinder;

b the total surface area of the cylinder.

10 A cone and cylinder as shown in the figure below have the same base area and the same volume. Find the height of the cone if the height of the cylinder is 30cm.



11 A soap company sells laundry detergents in two different containers with dimensions in the following figure. Which container holds more detergent? Justify your answer.



# Unit 8

# INTRODUCTION TO PROBABILITY

## Learning outcomes:

**After completing this unit, you will be able to:**

- understand the concept of probability;
- identify event, sample space and possible outcomes;
- find probability of simple event;
- apply the concept of probabilities in solving real – life problems.

## *Key terms*

- certain event
- equally likely
- event
- experiment
- impossible outcome
- likely event
- probability of an event
- sample space
- outcome
- unlikely event

## Introduction

We have been using the concept of probability in our daily life. For example, the probability of winning a game is high or low, there is high probability that it will rain today, etc, are used in our day to day activities. In these sentences, the word probability describes the estimates of the possibilities or chances to represent how likely or unlikely an event may happen.

Probability is being used by governments, economists, medical researchers and many other to predict the future by studying what has already happened.

In this unit, you will learn about introductory concepts of probability.

The first inquiry into the science of probability was made by an Italian physician and mathematician Girolamo Cardano (1501-1576). He predicated the date of his own death.

## 8.1 The Concept of Probability

### Activity 8.1

1 Identify each of the following situations or events as “impossible”, “possible” and “certain” to happen.

a A sun rises in the East.

b July is always after June.

c A newly born baby will be a girl.

d A flower can talk.

e It will rain tomorrow.

2 Describe the following situations or events as “likely” or “unlikely”

a It will rain in Juanary in Bahr Dar.

b All of the Grade 8 students in the counrty will pass the national examination

c You can complete highschool at the age of 18.

d There will be a heavy rain with the possibility of thunder in August

e All the students in your class have the same weight.

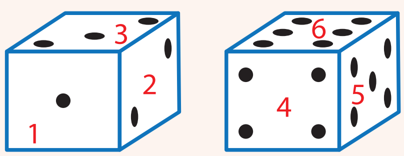
From your responses in Activity 8.1, observe that there are situations or events that are

- certain to happen; for example, it is certain that the sun rises in the East;
- possible to happen, for example, it may rain tomorrow or
- impossible to happen, for example, a flower can’t talk.

There are situations or events that are possible to happen, but it can be classified as likely and unlikely to happen. For example,

- most of the students complete their high school at the age of 18 and the situation that a you can complete high school at the age of 18 is a likely situation.
- The rainy season in Bahr Dar is from July to September and it is unlikely to rain in January. Therefore, the situation that it will rain in January in Bahr Dar is an unlikely situation.

Consider following games and the possible outcomes.

| **Situation** | **Possible Outcomes** |
|---|---|
| Tossing (flipping) a coin and observe the above face of the coin.<br><br>T H<br>Figure 8.1. Ethiopia coin | The possible chance is getting either a head (H) or a tail (T), but not both at a time<br>In every tossing, there is one chance out of the two possibilities, H or T. |
| Rolling (tossing) a fair (balanced) die and observing a number on the top face of the die.<br><br>Figure 8.2. Fair die | The possible chance is getting one of the numbers from the set {1, 2, 3, 4, 5, 6).<br>In every rolling, there is one chance out of the six possibilities, 1, 2, 3, 4, 5 or 6. The plural for of die is dice. |
| Selecting(picking) a card from a shuffled 52 cards<br><br>Figure 8.3. Pack of playing cards | The possible chance is getting one of the cards from the set of 52 cards marked by numbers from 2 to 10, letters J, Q, K and A.<br>There are four chances for each of the numbers from 2 to 10 or the letters J, Q, K and A out of the 52 possibilities. |

## Example 8.1

When you roll or toss a fair die once, the possible chance is getting one of the numbers from the set{1,2,3,4,5,6}. Classify the following situations as possible to happen, impossible to happen or certain to happen:

a getting number 7,

b getting an even number and

c getting a number less than 7.

**Solution**

a The situation of getting number 7 is impossible to happen, because, in rolling a die, the only numbers that will appear are only.1, 2, 3, 4, 5 or 6.

b The situation of getting an even number is possible to happen, because, in rolling a die, you can get even numbers 2, 4 or 6.

c The situation of getting number less than 7 is certain to happen, because, in rolling a die, the number that will appear is one of 1, 2, 3, 4, 5 or 6 and all of them are less than 7.

## Basic Terminologies in Probability

### Definition 8.1

A process of observation or measurement that produces quantifiable results is called an **experiment** and every observation of an experiment is called a **trial**.

### Example 8.2

Throwing a die, tossing a coin, drawing a card and spinning spinner are all trials, whereas, making each of the trials repeatedly is an experiment.

### Definition 8.2

The set of all possible outcomes of an experiment is called **the sample space** or the **probability set** of that experiment and it is usually denoted by S.

### Example 8.3

Find the sample space of each of the following experiments.

a Tossing a coin

b Rolling a die

**Solution**

a In tossing a coin, there are two possible outcomes, Head (H) and Tail (T). Therefore, the simple space of the experiment is S = {H, T}.

b In rolling a die, the numbers that will appear are 1, 2, 3, 4, 5 or 6. Therefore, the sample space of the experiment is S = {1,2,3,4,5,6}.

### Definition 8.3

**An outcome or a sample point** is a single result from an observation or a measurement. Outcome is a subset of a sample space.

## Example 8.4

Find the sample points of each of the following experiments.

a Tossing a coin

b Rolling a die

**Solution**

a In tossing a coin once, the sample space S = {H, T} has two outcomes {H} and {T}, which are sample points of S.

b In rolling a die, the sample space S = {1,2,3,4,5,6} has six outcomes {1}, {2}, {3}, {4}, {5} and {6}, which are sample points of S.

### Definition 8.4

**An Event** of a certain experiment is a set of one or more outcomes from the sample space and it is a subset of the sample space.

## Example 8.5

Find each of the following events of rolling a die;

a getting an odd number;

b getting an even number;

c getting a prime number;

d getting a number less than 6

**Solution**

The sample space of rolling a die is S ={1,2,3,4,5,6}.

a The odd numbers in the sample space are 1, 3 and 5.

Thus, the event of getting an odd number is {1,3,5}.

b The even numbers in the sample space are 2,4 and 6.

Therefore, the event set of getting an even number is {2,4,6}.

c The prime numbers in the sample space are 2,3 and 5.

Therefore, the event set of getting an even number is {2,3,5}.

d The numbers that are less than 6 in the sample space are 1, 2, 3, 4, 5.

Thus, the event of getting a number less that 6 is {1,2,3,4,5}.

### Definition 8.5

An event that does not have any chance of occurrence is called **impossible outcome.** An impossible outcome has no member.

## Example 8.6

The whole numbers 1 to 10 are written on equal sized piece of papers and placed in a box to drawn a lottery of students' prize. What is the chance of getting 12?

### Solution

There are no ways of getting 12 in this experiment. So, the chance of getting 12 is impossible. Therefore, this event is impossible with no sample point.

### Definition 8.6

If it is certain that an event will occur in an experiment, then it is called a **certain** or a **sure** event.

## Example 8.7

In rolling a die, what is the chance of getting a number less than 7?

### Solution

If you roll a die, the number that will appear is either 1, 2, 3, 4, 5 or 6 and all of the numbers are less than 7.

Thus, the event of getting a number less than 7 is a certain event.

## Example 8.8

There are 40 students in a class and all of them are female. From this class a teacher wants to select one student at random. What is the chance of selecting a female student?

### Solution

All the students in the class are female and if the teacher selects one student from the class, then that student will definitely be a female student.

Thus, the event of selecting a female sudent in the class is a certain event.

## Activity 8.2

Consider the following experiments:

a Tossing (flipping) a coin

b Rolling (tossing) a die

c Selecting a ball from a box containing 4 red big balls and 3 yellow small balls.

In which of the experiments does each sample point have an equal chance of occurrence? Give your reasons.

From your responses in Activity 8.2, observe that there are experiments in which all the the sample points have equal chance of occurrence and there are experiments that the sample points do not have equal chance of occurrence.

### Definition 8.7

Outcomes in a sample space of a given experiment are said to be **equally likely** if each outcome has equal chance of occurrence. Otherwise, it is **not equally likely** event.

## Example 8.9

In an experiment of rolling a six-faced die, if all the six faces have the same size and shape, then the six outcomes $\{1\},\{2\},\{3\},\{4\},\{5\}$ and $\{6\}$ have equal chance of occurrence. In this case, the event containing such types of outcomes is said to be equally likely event.

## Example 8.10

The spinner shown in Figure 8.4 has four unequal sectors, so when the pointer stops spinning, there are four possible outcomes indicated by the arrow.

a What are the possible outcomes?

b Give an example of an impossible outcome.

c Is the chance of getting banana and tomato equal? Why?

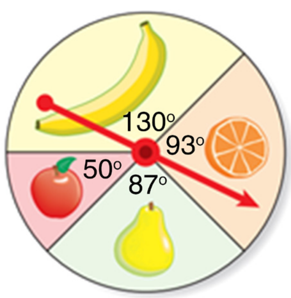



Figure 8.4. A spinner

### Solution

The spinner has four sectors with different sizes as shown in Figure 8.4. The sector of Banana has a central angle of 130°, the center for Orange has a central angle of 93°, the sector for Avocado has a central angle of 87°, the sector of Tomato has a central angle of 50°.

a When spinning the spinner, the pointer will stop at any of the four regions labeled by Orange, Banana, Tomato and Avocado. Therefore, the only possible outcomes of this experiment are elements of the following set.

S={Orange, Banana, Tomato, Avocado}

b The arrow cannot indicate on Lemon as there is no region labeled by Lemon. Hence {Lemon} is an example of impossible event.

Hence you can have other examples of impossible events

c Since the sector with Banana is much larger than the sector with Tomato, Banana has a high chance of occurrence than Tomato. Similarly, Banana has a high chance of occurrence than Orange and it has also a high chance of occurrence than Avocado.

Hence, the outcomes of this experiment are not equally likely outcomes.

### Multiple Events

## Activity 8.3

1 Suppose you toss a pair of coins at a time:

a list all possible outcomes of the experiment;

b how many possible outcomes are there?

2 Suppose you roll a pair of dice at the same time:

a list all possible outcomes of the experiment;

b how many possible outcomes are there?

From your responses in Activity 8.3, observe that the number of possible outcomes of a given experiment is obtained by multiplying the number of possible outcomes in each step.

## Note

If there are $m$ different ways that one event can occur and following this event there are $n$ different ways that another event can occur, then there are $m \times n$ different ways that both events can occur one after the other.

If there are $m$ different ways that one event can occur, following the first event there are $n$ different ways that the second event can occur and following the second event there are $k$ ways that the third event can occur, then there are $m \times n \times k$ different ways that all the three events can occur one after the other.

### Example 8.11

Determine the sample space and the number of elements in the sample space when you toss a coin two times.

### Solution

Since tossing a coin has two possible outcomes {H} or {T}, the number of sample points or outcomes in the sample space *S* when tossing a coin two times are HH, HT, TH and HH.

In the first throw there are two ways and in the second throw after the first throw there are two ways.

Therefore, the sample space is {HH, HT, TH, HH} and number of elements in the sample space is given by $2 \times 2 = 4$.

## Example 8.12

Determine the number of elements in the sample space when you toss a coin and roll a die.

### Solution

In tossing a coin there are two possible outcomes {H} or {T} and in rolling a die there six possible outcomes, 1, 2, 3, 4, 5 or 6.

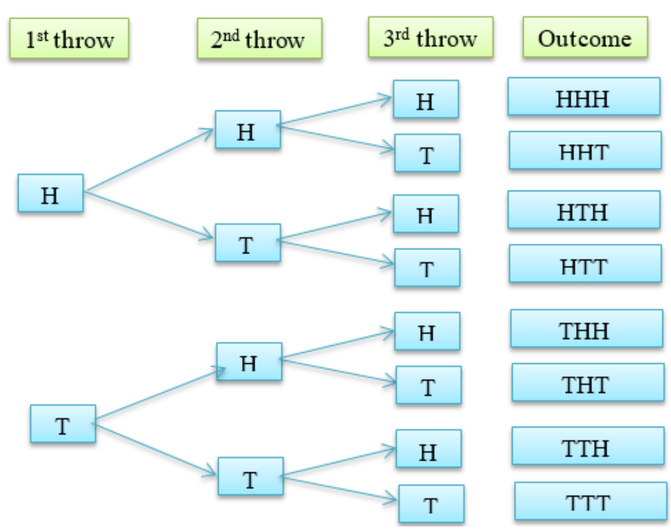
Therefore, the sample space has $2 \times 6 = 12$ elements.

## Example 8.13

Determine the sample space and the number of elements in the sample space when you toss a coin 3 times.

### Solution

Since a coin has two possible outcomes {H} or {T}, the number of sample points or outcomes in the sample space *S* when tossing a coin three times are listed in the tree diagram as follows.




That is the first event (throw) can happen in 2 ways, the second event (throw) can happen in 2 ways and the third event (throw) can happen in 2 ways.

Therefore, the number of elements in the sample space is given by $2 \times 2 \times 2 = 8$.

## Example 8.14

Find the number of elements in the sample space of rolling a die three times.

**Solution**

Rolling die has six possible outcomes {1}, {2}, {3}, {4}, {5} and {6} and if you roll (or throw) a die three times, the number of sample points or outcomes in the sample space is $6 \times 6 \times 6 = 216$.

## Exercise 8.1

1 Determine the sample space and sample points for each of the following experiments.

 a Tossing five coins at one time and observing the sequence of heads and tails.

 b Rolling a die and tossing a coin

 c Answering a true or false question.

 d Answering a multiple choice question with four alternatives A, B, C and D.

2 Let each number 1 to 20 be written on one of 20 equal sized cards. If one card is chosen at random, then list the elements of the following events.

 a The number is less than 5

 b The number is between 6 and 15

 c The number is greater than 21

 d The number is a prime number.

3 List all possible outcomes of an event if two dice are rolled at a time

 a the same numbers are shown on both dice

 b the product of the two numbers is 1

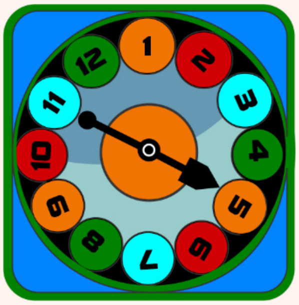
 c the sum of the numbers is 13

 d a prime number is shown on each die

4 Consider the Spinner Board Game as in the right.

 a List the possible outcomes of spinning the spinner.

 b How many outcomes include landing on an even number?



## 8.2 Probability of Simple Event

### Activity 8.4

Do the following activity in a group of two or three students.

Toss (flip) a coin 10 times and record the results in a table.

a How many times did you get head?

b How many times did you get tail?

c Write the number of heads as a fraction of the total number of tosses.

d Write the number of tails as a fraction of the total number of tosses.

From your responses in Activity 8.4, observe that, the chance of occurrence of a given event is the ratio of the number of elements in each event to the number of elements in the sample space. This ratio is called the probability of the event.

### Definition 8.8

In an experiment, if all outcomes are equally likely, then the probability of a simple event E, P(E), is the ratio of the number of successful outcomes in the event "*E*", n(E) to the total number of possible outcomes in the sample space "*S*", n(S). That is,

$$P(E)=\frac{\text{Number of elements of event E}}{\text{Number of elements of the sample space S}}=\frac{n(E)}{n(S)}$$

### Example 8.15

Suppose you toss a coin. Find the probability of getting:

a a head

b a tail

**Solution**

The possible outcomes are head (H) and tail (T).

Then, the sample space is given by S={H,T}.

a the event of getting head is E={H}.

Thus the probability of getting head is $P(E)=\frac{n(E)}{n(S)}=\frac{1}{2}$.

b the event of getting tail is E={T}.

Thus the probability of getting a tail is $P(E)=\frac{n(E)}{n(S)}=\frac{1}{2}$.

## Example 8.16

Suppose you roll a die. Then determine the probability of getting

a a composite number.

b a prime number

c a number less than 6

d a number divisible by 5.

### Solution

The possible outcomes are 1,2,3,4,5 and 6. Then, the sample space is given by

$$S=\{1,2,3,4,5,6\} \text{ and } n(S)=6.$$

a Let E be the event for getting a composite number. Then E= $\{4,6\}$ and $n(E)=2$.

Therefore, $P(E)=\frac{n(E)}{n(S)}=\frac{2}{6}=\frac{1}{3}$.

b The event of getting a prime number is E = {2, 3, 5} with n (E) = 3

Therefore, $P(E)=\frac{n(E)}{n(S)}=\frac{3}{6}=\frac{1}{3}$

c The event of getting a number less than 6 is E = {1, 2, 3, 4, 5} and n (E) = 5.

Thus, $P(E)=\frac{n(E)}{n(S)}=\frac{5}{6}$.

d The event of getting a number divisible by 5 is E= (5) and n (E)=1

So, $P(E)=\frac{n(E)}{n(S)}=\frac{1}{6}$.

## Example 8.17

Two fair dice are rolled. Find the probability of each of the following events.

a An event of getting two numbers whose sum is 7

b An event of getting two numbers whose sum is 13

c An event of getting two numbers whose sum is less than 13

### Solution

There are 6 numbers on each die. The sample space has 6 × 6 = 36 outcomes. It is illustrated as follows

| | | **Second die** | | | | | |
|---|---|---|---|---|---|---|---|
| | | **1** | **2** | **3** | **4** | **5** | **6** |
| **First Die** | **1** | (1,1) | (1,2) | (1,3) | (1,4) | (1,5) | (1,6) |
| | **2** | (2,1) | (2,2) | (2,3) | (2,4) | (2,5) | (2,6) |
| | **3** | (3,1) | (3,2) | (3,3) | (3,4) | (3,5) | (3,6) |
| | **4** | (4,1) | (4,2) | (4,3) | (4,4) | (4,5) | (4,6) |
| | **5** | (5,1) | (5,2) | (5,3) | (5,4) | (5,5) | (5,6) |
| | **6** | (6,1) | (6,2) | (6,3) | (6,4) | (6,5) | (6,6) |

a The possible outcomes of numbers with sum 7 are {(1, 6), (2, 5), (3, 4), (4, 3), (5, 2) (6, 1)}. Hence there are 6 possible outcomes.

$$P(\text{sum } is\,7) = \frac{\text{number of sucessful outcomes}}{\text{total numberof possible outcomes}} = \frac{6}{36} = \frac{1}{6}$$

b There are no outcome with sum is 13. Thus, this is an impossible event.

$$P(sum\ is\ 13) = \frac{0}{36} = 0$$

c All the 36 outcomes have sum less than 13. Thus, this event is certain. $P(sum < 13) = \frac{36}{36} = 1$.

## Activity 8.5

Suppose you toss a coin. Then find the probability getting:

a a head.

b number less than 3.

c a tail

d a head or a tail.

From your respponses in Activity 8.5, observe that the probability of an event is always between 0 and 1.

## Note

1 If P(E) is the probability of a given event E, then $0 \leq P(E) \leq 1$.

2 In an experiment, if an event is likely to occur, then the probability of an event is close to 1, where as if it is not likely to occur, then the probability of the event is close to 0.

| Event, $E$ | Impossible | Unlikely (Poor chance) | Even chance | Likely (Good chance) | Certain |
|---|---|---|---|---|---|
| Probability scale | $P(E)=0$ | $0<P(E)<\frac{1}{2}$ | $P(E)=\frac{1}{2}$ | $\frac{1}{2}<P(E)<1$ | $P(E)=1$ |

## Example 8.18

Consider the following list of numbers, 7, 12, 15, 23, 32, 33, 34, 50, 65. If one of these numbers is selected at random, then what is the probability that the selected number is

a a number less than 6?

b an odd number?

c a prime number?

d a number less than 100?

### Solution

In this experiment, there are 9 sample points.

a Out of the nine numbers given, none of them is less than 6. So, the probability of selecting a number that is less than 6 is $\frac{0}{9}=0$.

Therefore, the event of selecting a number less than 6 is impossible to occur.

b Out of the nine given numbers, only five of them are odd; 7, 15, 23, 33 and 65. So, the probability of selecting an odd number is $\frac{5}{9}=0.555\cdots$, which is between $\frac{1}{2}$ and 1.

Therefore, the event of selecting an odd number is likely (good chance) to occur.

c Out of the nine numbers given, only two of them are primes; 7 and 23. So, the probability of selecting a prime number is $\frac{2}{9}=0.222\cdots$, which is between 0 and $\frac{1}{2}$

Therefore, the event of selecting a prime number is unlikely (poor chance) to occur.

d All of the nine numbers are less than 100. So, the probability of selecting a number less than 100 is $\frac{9}{9}=1$.

Therefore, the event of selecting a number less than 100 is certain to occur.

## Note

In a probability experiment or trial, a particular result, say event $A$ must either happen or does not happen. Thus,

$$P(\text{A does happen})+P(\text{A does not happen})=1$$

This leads to a very useful rule that:

$$P(\text{A does not happen})=1-P(\text{A does happen})$$

## Example 8.19

If the probability that it will rain on Easter day is 0.15, then find the probability that it will not rain on Easter day.

### Solution

Let $E$ be an event that it will rain on Easter day and $E^C$ be an event that it will not rain on Easter day.

$$P(E)+P(E^c)=1$$

$$P(E^c)=1-P(E)=1-0.15=0.85$$

Therefore, the probability that it will not rain on Easter day is 0.85

## Exercise 8.2

1 A letter is chosen at random from the word "PROBABILITY". Find the probability that the chosen letter is:

a R
b B
c I or A.
d not a vowel

2 A natural number less than 30 is chosen at random. Find the probability that the chosen number is:

a a multiple of 4
b a perfect square
c a prime
d a factor of 36
e a perfect cube
f Less than 6

3 Two dice are rolled at the same time. Find the probability of each of the following events.

a The sum of the numbers is prime.
b The sum of the numbers is less than 9.
c The sum of the numbers is greater than 7.
d The sum of the numbers is greater than 5 and less than 10.

4 If the probability of winning a specific game is 0.3, what is the probability of losing that game, where win or loss are the only options of the game?

5 A jar contains 54 marbles of the same size which are blue, green or white in color. The probability of selecting a blue marble at random is $\frac{1}{3}$ and the probability of selecting a green marble at random is $\frac{4}{9}$ .

a Find the numbers of blue, green and white marbles in the jar.
b What is the probability of selecting a white marble?

6 In certain shop there are 600 electric bulbs in a box where 12 are defective bulbs. You bought one bulb taken out at random from the box.

a What is the probability that the bulb is defective?
b What is the probability that the bulb is non-defective?

7 A bag contains 8 identical discs of which four of them are red in color, three of them are blue in color and one of them is yellow in color. If one disc is drawn at random, then find the probability that drawn card is:

a red
b blue
c yellow or blue
d not blue

8 If a city has a population of 1,000,000 and the death rate in car accidents is 500 per year, then what is the chance of an individual living in the city dying by a car accident in a certain year?

9 A School Music Band is selling lottery tickets for community service. They sold 3,000 tickets at 100 Birr each. Your parents spent 2000 Birr on the lottery tickets.

a What is the probability that your parents will win the lottery?

b What is the probability that your parents will not win the lottery?

# Unit Summary

1 Experiment: An activity involving chance in which results are observed.

2 Outcome: A single possible result of an experiment or trial.

3 Sample space: The set of all possible outcomes of an experiment.

4 Sample point: An element of the sample space

5 Event: The set of one or more outcomes from sample space.

6 Probability: The chance that something will happen or will not happen.

7 Probability of an event $E$ is calculated as

$$P(E)=\frac{\text{Number of possible outcomes in an event E}}{\text{Number of all outcomes in the sample space S}}$$

8 An event can be categorized as Impossible event, Certain event, Likely event or Unlikely event

9 Probability scale

$P(\text{impossible event})=0$ $\quad 0<P(\text{Unlikely event})<\frac{1}{2}$

$P(\text{Certain event})=1$ $\quad \frac{1}{2}<P(\text{Likely event})<1$

10 In a fair experiment, each outcome has an equal chance to occur and hence is equally likely.

11 For any event $E$,

a $0\leq P(E)\leq 1$    b $P(E)+P(not\ E)=1$

## Review Exercises

1 Three dice are rolled at the same time. Find the probability of each of the following events.

a the sum of the numbers is 5

b the product the numbers is 12

c the sum of the numbers is 517

2 A number is selected at random from whole numbers 1 to 100. Find the probability that the selected number is:

a an odd number

b the number is even or divisible by 5

c the number is divisible by 5

d the number is divisible by 3 or 5.

3 One letter is chosen at random from the word "ISOSCELES". Find the probability of choosing

a letter C

b a vowel letter

c letter E

d a consonant letter

4 There are 52 cards in a deck (playing cards). The playing cards have 4 groups; each group has 13 members of Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King. What is the probability of drawing a Queen card at random?

5 A school sold 348 lottery tickets to raise money to buy equipments for a playground. Jemal bought 5 tickets. How likely is Jemal winnig the lottery?

6 There are 54 Ethiopian and 17 foreigners in a meeting hall. One person is chosen at random as a chairperson of the meeting. Find the probability that the chairperson is:

a an Ethiopian

b a foreigner

c either an Ethiopian or a foreigner

7 The table below shows students distribution per grade in a school.

| | **Grades** | | | | | | | | **Total** |
|---|---|---|---|---|---|---|---|---|---|
| **Sex** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | |
| **M** | 33 | 18 | 20 | 23 | 17 | 16 | 25 | 30 | **182** |
| **F** | 17 | 12 | 20 | 19 | 21 | 34 | 15 | 20 | **158** |
| **Total** | **50** | **30** | **40** | **42** | **38** | **50** | **40** | **50** | **340** |

If a student is selected at random as chairperon of students' association of the school, then find the probability that the chairperson is:

a a female student from grade 6

b a male student either from grade 5 or grade 7

8 In "Ekub" association, there are 98 members. Each member pays 2000 Birr per week. W/ro. Fatuma is one of the members who had not received her "Ekub" until the $46^{th}$ "Ekub". What is the probability Wro. Fatuma gets the "Ekub" in $46^{th}$ "Ekub"?

**Table A-1:** -table, for $1.0 \leq x \leq 5.55$

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.0 | 1.000 | 1.020 | 1.040 | 1.061 | 1.082 | 1.103 | 1.124 | 1.145 | 1.166 | 1.188 |
| 1.1 | 1.210 | 1.232 | 1.254 | 1.277 | 1.300 | 1.323 | 1.346 | 1.369 | 1.392 | 1.416 |
| 1.2 | 1.440 | 1.464 | 1.488 | 1.513 | 1.538 | 1.563 | 1.588 | 1.613 | 1.638 | 1.664 |
| 1.3 | 1.690 | 1.716 | 1.742 | 1.769 | 1.796 | 1.823 | 1.850 | 1.877 | 1.904 | 1.932 |
| 1.4 | 1.960 | 1.988 | 2.016 | 2.045 | 2.074 | 2.103 | 2.132 | 2.161 | 2.190 | 2.220 |
| 1.5 | 2.250 | 2.280 | 2.310 | 2.341 | 2.372 | 2.403 | 2.434 | 2.465 | 2.496 | 2.528 |
| 1.6 | 2.560 | 2.592 | 2.624 | 2.657 | 2.690 | 2.723 | 2.756 | 2.789 | 2.822 | 2.856 |
| 1.7 | 2.890 | 2.924 | 2.958 | 2.993 | 3.028 | 3.063 | 3.098 | 3.133 | 3.168 | 3.204 |
| 1.8 | 3.240 | 3.276 | 3.312 | 3.349 | 3.386 | 3.423 | 3.460 | 3.497 | 3.534 | 3.572 |
| 1.9 | 3.610 | 3.648 | 3.686 | 3.725 | 3.764 | 3.803 | 3.842 | 3.881 | 3.920 | 3.960 |
| 2.0 | 4.000 | 4.040 | 4.080 | 4.121 | 4.162 | 4.203 | 4.244 | 4.285 | 4.326 | 4.368 |
| 2.1 | 4.410 | 4.452 | 4.494 | 4.537 | 4.580 | 4.623 | 4.666 | 4.709 | 4.752 | 4.796 |
| 2.2 | 4.840 | 4.884 | 4.928 | 4.973 | 5.018 | 5.063 | 5.108 | 5.153 | 5.198 | 5.244 |
| 2.3 | 5.290 | 5.336 | 5.382 | 5.429 | 5.476 | 5.523 | 5.570 | 5.617 | 5.664 | 5.712 |
| 2.4 | 5.760 | 5.808 | 5.856 | 5.905 | 5.954 | 6.003 | 6.052 | 6.101 | 6.150 | 6.200 |
| 2.5 | 6.250 | 6.300 | 6.350 | 6.401 | 6.452 | 6.503 | 6.554 | 6.605 | 6.656 | 6.708 |
| 2.6 | 6.760 | 6.812 | 6.864 | 6.917 | 6.970 | 7.023 | 7.076 | 7.129 | 7.182 | 7.236 |
| 2.7 | 7.290 | 7.344 | 7.398 | 7.453 | 7.508 | 7.563 | 7.618 | 7.673 | 7.728 | 7.784 |
| 2.8 | 7.840 | 7.896 | 7.952 | 8.009 | 8.066 | 8.123 | 8.180 | 8.237 | 8.294 | 8.352 |
| 2.9 | 8.410 | 8.468 | 8.526 | 8.585 | 8.644 | 8.703 | 8.762 | 8.821 | 8.880 | 8.940 |
| 3.0 | 9.000 | 9.060 | 9.120 | 9.181 | 9.242 | 9.303 | 9.364 | 9.425 | 9.486 | 9.548 |
| 3.1 | 9.610 | 9.672 | 9.734 | 9.797 | 9.860 | 9.923 | 9.986 | 10.049 | 10.112 | 10.176 |
| 3.2 | 10.240 | 10.304 | 10.368 | 10.433 | 10.498 | 10.563 | 10.628 | 10.693 | 10.758 | 10.824 |
| 3.3 | 10.890 | 10.956 | 11.022 | 11.089 | 11.156 | 11.223 | 11.290 | 11.357 | 11.424 | 11.492 |
| 3.4 | 11.560 | 11.628 | 11.696 | 11.765 | 11.834 | 11.903 | 11.972 | 12.041 | 12.110 | 12.180 |
| 3.5 | 12.250 | 12.320 | 12.390 | 12.461 | 12.532 | 12.603 | 12.674 | 12.745 | 12.816 | 12.888 |
| 3.6 | 12.960 | 13.032 | 13.104 | 13.177 | 13.250 | 13.323 | 13.396 | 13.469 | 13.542 | 13.616 |
| 3.7 | 13.690 | 13.764 | 13.838 | 13.913 | 13.988 | 14.063 | 14.138 | 14.213 | 14.288 | 14.364 |
| 3.8 | 14.440 | 14.516 | 14.592 | 14.669 | 14.746 | 14.823 | 14.900 | 14.977 | 15.054 | 15.132 |
| 3.9 | 15.210 | 15.288 | 15.366 | 15.445 | 15.524 | 15.603 | 15.682 | 15.761 | 15.840 | 15.920 |
| 4.0 | 16.000 | 16.080 | 16.160 | 16.241 | 16.322 | 16.403 | 16.484 | 16.565 | 16.646 | 16.728 |
| 4.1 | 16.810 | 16.892 | 16.974 | 17.057 | 17.140 | 17.223 | 17.306 | 17.389 | 17.472 | 17.556 |
| 4.2 | 17.640 | 17.724 | 17.808 | 17.893 | 17.978 | 18.063 | 18.148 | 18.233 | 18.318 | 18.404 |
| 4.3 | 18.490 | 18.576 | 18.662 | 18.749 | 18.836 | 18.923 | 19.010 | 19.097 | 19.184 | 19.272 |
| 4.4 | 19.360 | 19.448 | 19.536 | 19.625 | 19.714 | 19.803 | 19.892 | 19.981 | 20.070 | 20.160 |
| 4.5 | 20.250 | 20.340 | 20.430 | 20.521 | 20.612 | 20.703 | 20.794 | 20.885 | 20.976 | 21.068 |
| 4.6 | 21.160 | 21.252 | 21.344 | 21.437 | 21.530 | 21.623 | 21.716 | 21.809 | 21.902 | 21.996 |
| 4.7 | 22.090 | 22.184 | 22.278 | 22.373 | 22.468 | 22.563 | 22.658 | 22.753 | 22.848 | 22.944 |
| 4.8 | 23.040 | 23.136 | 23.232 | 23.329 | 23.426 | 23.523 | 23.620 | 23.717 | 23.814 | 23.912 |
| 4.9 | 24.010 | 24.108 | 24.206 | 24.305 | 24.404 | 24.503 | 24.602 | 24.701 | 24.800 | 24.900 |
| 5.0 | 25.000 | 25.100 | 25.200 | 25.301 | 25.402 | 25.503 | 25.604 | 25.705 | 25.806 | 25.908 |
| 5.1 | 26.010 | 26.112 | 26.214 | 26.317 | 26.420 | 26.523 | 26.626 | 26.729 | 26.832 | 26.936 |
| 5.2 | 27.040 | 27.144 | 27.248 | 27.353 | 27.458 | 27.563 | 27.668 | 27.773 | 27.878 | 27.984 |
| 5.3 | 28.090 | 28.196 | 28.302 | 28.409 | 28.516 | 28.623 | 28.730 | 28.837 | 28.944 | 29.052 |
| 5.4 | 29.160 | 29.268 | 29.376 | 29.485 | 29.594 | 29.703 | 29.812 | 29.921 | 30.030 | 30.140 |
| 5.5 | 30.250 | 30.360 | 30.470 | 30.581 | 30.692 | 30.803 | 30.914 | 31.025 | 31.136 | 31.248 |

$\sqrt{14.516} \approx 3.81$, $\sqrt{17.978} \approx 4.24$, $\sqrt{20.07} \approx 4.48$

**Table A-2:** $x^2$ -table, for $5.6 \leq x \leq 10.0$

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 5.6 | 31.360 | 31.472 | 31.584 | 31.697 | 31.810 | 31.923 | 32.036 | 32.149 | 32.262 | 32.376 |
| 5.7 | 32.490 | 32.604 | 32.718 | 32.833 | 32.948 | 33.063 | 33.178 | 33.293 | 33.408 | 33.524 |
| 5.8 | 33.640 | 33.756 | 33.872 | 33.989 | 34.106 | 34.223 | 34.340 | 34.457 | 34.574 | 34.692 |
| 5.9 | 34.810 | 34.928 | 35.046 | 35.165 | 35.284 | 35.403 | 35.522 | 35.641 | 35.760 | 35.880 |
| 6.0 | 36.000 | 36.120 | 36.24 | 36.361 | 36.482 | 36.603 | 36.724 | 36.845 | 36.966 | 37.088 |
| 6.1 | 37.210 | 37.332 | 37.454 | 37.577 | 37.700 | 37.823 | 37.946 | 38.069 | 38.192 | 38.316 |
| 6.2 | 38.440 | 38.564 | 38.688 | 38.813 | 38.938 | 39.063 | 39.188 | 39.313 | 39.438 | 39.564 |
| 6.3 | 39.690 | 39.816 | 39.942 | 40.069 | 40.196 | 40.323 | 40.450 | 40.577 | 40.704 | 40.832 |
| 6.4 | 40.960 | 41.088 | 41.216 | 41.345 | 41.474 | 41.603 | 41.732 | 41.861 | 41.990 | 42.120 |
| 6.5 | 42.250 | 42.380 | 42.510 | 42.641 | 42.772 | 42.903 | 43.034 | 43.165 | 43.296 | 43.428 |
| 6.6 | 43.560 | 43.692 | 43.824 | 43.957 | 44.09 | 44.223 | 44.356 | 44.489 | 44.622 | 44.756 |
| 6.7 | 44.890 | 45.024 | 45.158 | 45.293 | 45.428 | 45.563 | 45.698 | 45.833 | 45.968 | 46.104 |
| 6.8 | 46.240 | 46.376 | 46.512 | 46.649 | 46.786 | 46.923 | 47.060 | 47.197 | 47.334 | 47.472 |
| 6.9 | 47.610 | 47.748 | 47.886 | 48.025 | 48.164 | 48.303 | 48.442 | 48.581 | 48.720 | 48.860 |
| 7.0 | 49.000 | 49.140 | 49.280 | 49.421 | 49.562 | 49.703 | 49.844 | 49.985 | 50.126 | 50.268 |
| 7.1 | 50.410 | 50.552 | 50.694 | 50.837 | 50.980 | 51.123 | 51.266 | 51.409 | 51.552 | 51.696 |
| 7.2 | 51.840 | 51.984 | 52.128 | 52.273 | 52.418 | 52.563 | 52.708 | 52.853 | 52.998 | 53.144 |
| 7.3 | 53.290 | 53.436 | 53.582 | 53.729 | 53.876 | 54.023 | 54.170 | 54.317 | 54.464 | 54.612 |
| 7.4 | 54.760 | 54.908 | 55.056 | 55.205 | 55.354 | 55.503 | 55.652 | 55.801 | 55.950 | 56.100 |
| 7.5 | 56.250 | 56.400 | 56.55 | 56.701 | 56.852 | 57.003 | 57.154 | 57.305 | 57.456 | 57.608 |
| 7.6 | 57.760 | 57.912 | 58.064 | 58.217 | 58.370 | 58.523 | 58.676 | 58.829 | 58.982 | 59.136 |
| 7.7 | 59.290 | 59.444 | 59.598 | 59.753 | 59.908 | 60.063 | 60.218 | 60.373 | 60.528 | 60.684 |
| 7.8 | 60.840 | 60.996 | 61.152 | 61.309 | 61.466 | 61.623 | 61.780 | 61.937 | 62.094 | 62.252 |
| 7.9 | 62.410 | 62.568 | 62.726 | 62.885 | 63.044 | 63.203 | 63.362 | 63.521 | 63.680 | 63.84 |
| 8.0 | 64.000 | 64.160 | 64.320 | 64.481 | 64.642 | 64.803 | 64.964 | 65.125 | 65.286 | 65.448 |
| 8.1 | 65.610 | 65.772 | 65.934 | 66.097 | 66.260 | 66.423 | 66.586 | 66.749 | 66.912 | 67.076 |
| 8.2 | 67.240 | 67.404 | 67.568 | 67.733 | 67.898 | 68.063 | 68.228 | 68.393 | 68.558 | 68.724 |
| 8.3 | 68.890 | 69.056 | 69.222 | 69.389 | 69.556 | 69.723 | 69.890 | 70.057 | 70.224 | 70.392 |
| 8.4 | 70.560 | 70.728 | 70.896 | 71.065 | 71.234 | 71.403 | 71.572 | 71.741 | 71.910 | 72.080 |
| 8.5 | 72.250 | 72.420 | 72.590 | 72.761 | 72.932 | 73.103 | 73.274 | 73.445 | 73.616 | 73.788 |
| 8.6 | 73.960 | 74.132 | 74.304 | 74.477 | 74.650 | 74.823 | 74.996 | 75.169 | 75.342 | 75.516 |
| 8.7 | 75.690 | 75.864 | 76.038 | 76.213 | 76.388 | 76.563 | 76.738 | 76.913 | 77.088 | 77.264 |
| 8.8 | 77.440 | 77.616 | 77.792 | 77.969 | 78.146 | 78.323 | 78.500 | 78.677 | 78.854 | 79.032 |
| 8.9 | 79.210 | 79.388 | 79.566 | 79.745 | 79.924 | 80.103 | 80.282 | 80.461 | 80.640 | 80.820 |
| 9.0 | 81.000 | 81.180 | 81.360 | 81.541 | 81.722 | 81.903 | 82.084 | 82.265 | 82.446 | 82.628 |
| 9.1 | 82.810 | 82.992 | 83.174 | 83.357 | 83.540 | 83.723 | 83.906 | 84.089 | 84.272 | 84.456 |
| 9.2 | 84.640 | 84.824 | 85.008 | 85.193 | 85.378 | 85.563 | 85.748 | 85.933 | 86.118 | 86.304 |
| 9.3 | 86.490 | 86.676 | 86.862 | 87.049 | 87.236 | 87.423 | 87.610 | 87.797 | 87.984 | 88.172 |
| 9.4 | 88.360 | 88.548 | 88.736 | 88.925 | 89.114 | 89.303 | 89.492 | 89.681 | 89.870 | 90.060 |
| 9.5 | 90.250 | 90.440 | 90.630 | 90.821 | 91.012 | 91.203 | 91.394 | 91.585 | 91.776 | 91.968 |
| 9.6 | 92.160 | 92.352 | 92.544 | 92.737 | 92.930 | 93.123 | 93.316 | 93.509 | 93.702 | 93.896 |
| 9.7 | 94.090 | 94.284 | 94.478 | 94.673 | 94.868 | 95.063 | 95.258 | 95.453 | 95.648 | 95.844 |
| 9.8 | 96.040 | 96.236 | 96.432 | 96.629 | 96.826 | 97.023 | 97.220 | 97.417 | 97.614 | 97.812 |
| 9.9 | 98.010 | 98.208 | 98.406 | 98.605 | 98.804 | 99.003 | 99.202 | 99.401 | 99.600 | 99.800 |
| 10.0 | 100.000 | 100.200 | 100.400 | 100.601 | 100.802 | 101.003 | 101.204 | 101.405 | 101.606 | 101.808 |

$\sqrt{34.928} \approx 5.91$, $\sqrt{83.54} \approx 9.14$, $\sqrt{101.808} \approx 10.09$

**Table B-1**: $x^3$-table, for $1.0 \leq x \leq 5.55$

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.0 | 1.000 | 1.030 | 1.061 | 1.093 | 1.125 | 1.158 | 1.191 | 1.225 | 1.260 | 1.295 |
| 1.1 | 1.331 | 1.368 | 1.405 | 1.443 | 1.482 | 1.521 | 1.561 | 1.602 | 1.643 | 1.685 |
| 1.2 | 1.728 | 1.772 | 1.816 | 1.861 | 1.907 | 1.953 | 2.000 | 2.048 | 2.097 | 2.147 |
| 1.3 | 2.197 | 2.248 | 2.300 | 2.353 | 2.406 | 2.460 | 2.515 | 2.571 | 2.628 | 2.686 |
| 1.4 | 2.744 | 2.803 | 2.863 | 2.924 | 2.986 | 3.049 | 3.112 | 3.177 | 3.242 | 3.308 |
| 1.5 | 3.375 | 3.443 | 3.512 | 3.582 | 3.652 | 3.724 | 3.796 | 3.870 | 3.944 | 4.020 |
| 1.6 | 4.096 | 4.173 | 4.252 | 4.331 | 4.411 | 4.492 | 4.574 | 4.657 | 4.742 | 4.827 |
| 1.7 | 4.913 | 5.000 | 5.088 | 5.178 | 5.268 | 5.359 | 5.452 | 5.545 | 5.640 | 5.735 |
| 1.8 | 5.832 | 5.93 | 6.029 | 6.128 | 6.230 | 6.332 | 6.435 | 6.539 | 6.645 | 6.751 |
| 1.9 | 6.859 | 6.968 | 7.078 | 7.189 | 7.301 | 7.415 | 7.530 | 7.645 | 7.762 | 7.881 |
| 2.0 | 8.000 | 8.121 | 8.242 | 8.365 | 8.490 | 8.615 | 8.742 | 8.870 | 8.999 | 9.129 |
| 2.1 | 9.261 | 9.394 | 9.528 | 9.664 | 9.800 | 9.938 | 10.078 | 10.218 | 10.360 | 10.503 |
| 2.2 | 10.648 | 10.794 | 10.941 | 11.09 | 11.239 | 11.391 | 11.543 | 11.697 | 11.852 | 12.009 |
| 2.3 | 12.167 | 12.326 | 12.487 | 12.649 | 12.813 | 12.978 | 13.144 | 13.312 | 13.481 | 13.652 |
| 2.4 | 13.824 | 13.998 | 14.172 | 14.349 | 14.527 | 14.706 | 14.887 | 15.069 | 15.253 | 15.438 |
| 2.5 | 15.625 | 15.813 | 16.003 | 16.194 | 16.387 | 16.581 | 16.777 | 16.975 | 17.174 | 17.374 |
| 2.6 | 17.576 | 17.78 | 17.985 | 18.191 | 18.400 | 18.610 | 18.821 | 19.034 | 19.249 | 19.465 |
| 2.7 | 19.683 | 19.903 | 20.124 | 20.346 | 20.571 | 20.797 | 21.025 | 21.254 | 21.485 | 21.718 |
| 2.8 | 21.952 | 22.188 | 22.426 | 22.665 | 22.906 | 23.149 | 23.394 | 23.640 | 23.888 | 24.138 |
| 2.9 | 24.389 | 24.642 | 24.897 | 25.154 | 25.412 | 25.672 | 25.934 | 26.198 | 26.464 | 26.731 |
| 3.0 | 27.000 | 27.271 | 27.544 | 27.818 | 28.094 | 28.373 | 28.653 | 28.934 | 29.218 | 29.504 |
| 3.1 | 29.791 | 30.080 | 30.371 | 30.664 | 30.959 | 31.256 | 31.554 | 31.855 | 32.157 | 32.462 |
| 3.2 | 32.768 | 33.076 | 33.386 | 33.698 | 34.012 | 34.328 | 34.646 | 34.966 | 35.288 | 35.611 |
| 3.3 | 35.937 | 36.265 | 36.594 | 36.926 | 37.260 | 37.595 | 37.933 | 38.273 | 38.614 | 38.958 |
| 3.4 | 39.304 | 39.652 | 40.002 | 40.354 | 40.708 | 41.064 | 41.422 | 41.782 | 42.144 | 42.509 |
| 3.5 | 42.875 | 43.244 | 43.614 | 43.987 | 44.362 | 44.739 | 45.118 | 45.499 | 45.883 | 46.268 |
| 3.6 | 46.656 | 47.046 | 47.438 | 47.832 | 48.229 | 48.627 | 49.028 | 49.431 | 49.836 | 50.243 |
| 3.7 | 50.653 | 51.065 | 51.479 | 51.895 | 52.314 | 52.734 | 53.157 | 53.583 | 54.010 | 54.440 |
| 3.8 | 54.872 | 55.306 | 55.743 | 56.182 | 56.623 | 57.067 | 57.512 | 57.961 | 58.411 | 58.864 |
| 3.9 | 59.319 | 59.776 | 60.236 | 60.698 | 61.163 | 61.630 | 62.099 | 62.571 | 63.045 | 63.521 |
| 4.0 | 64.000 | 64.481 | 64.965 | 65.451 | 65.939 | 66.430 | 66.923 | 67.419 | 67.917 | 68.418 |
| 4.1 | 68.921 | 69.427 | 69.935 | 70.445 | 70.958 | 71.473 | 71.991 | 72.512 | 73.035 | 73.560 |
| 4.2 | 74.088 | 74.618 | 75.151 | 75.687 | 76.225 | 76.766 | 77.309 | 77.854 | 78.403 | 78.954 |
| 4.3 | 79.507 | 80.063 | 80.622 | 81.183 | 81.747 | 82.313 | 82.882 | 83.453 | 84.028 | 84.605 |
| 4.4 | 85.184 | 85.766 | 86.351 | 86.938 | 87.528 | 88.121 | 88.717 | 89.315 | 89.915 | 90.519 |
| 4.5 | 91.125 | 91.734 | 92.345 | 92.960 | 93.577 | 94.196 | 94.819 | 95.444 | 96.072 | 96.703 |
| 4.6 | 97.336 | 97.972 | 98.611 | 99.253 | 99.897 | 100.545 | 101.195 | 101.848 | 102.503 | 103.162 |
| 4.7 | 103.823 | 104.487 | 105.154 | 105.824 | 106.496 | 107.172 | 107.85 | 108.531 | 109.215 | 109.902 |
| 4.8 | 110.592 | 111.285 | 111.98 | 112.679 | 113.38 | 114.084 | 114.791 | 115.501 | 116.214 | 116.930 |
| 4.9 | 117.649 | 118.371 | 119.095 | 119.823 | 120.554 | 121.287 | 122.024 | 122.763 | 123.506 | 124.251 |
| 5.0 | 125.000 | 125.752 | 126.506 | 127.264 | 128.024 | 128.788 | 129.554 | 130.324 | 131.097 | 131.872 |
| 5.1 | 132.651 | 133.433 | 134.218 | 135.006 | 135.797 | 136.591 | 137.388 | 138.188 | 138.992 | 139.798 |
| 5.2 | 140.608 | 141.421 | 142.237 | 143.056 | 143.878 | 144.703 | 145.532 | 146.363 | 147.198 | 148.036 |
| 5.3 | 148.877 | 149.721 | 150.569 | 151.419 | 152.273 | 153.130 | 153.991 | 154.854 | 155.721 | 156.591 |
| 5.4 | 157.464 | 158.34 | 159.220 | 160.103 | 160.989 | 161.879 | 162.771 | 163.667 | 164.567 | 165.469 |
| 5.5 | 166.375 | 167.284 | 168.197 | 169.112 | 170.031 | 170.954 | 171.880 | 172.809 | 173.741 | 174.677 |

$\sqrt[3]{19.683} \approx 2.70$, $\sqrt[3]{65.451} \approx 4.03$, $\sqrt[3]{123.506} \approx 4.98$

**Table B-2:** $x^3$-table, for $5.6 \leq x \leq 10.0$

| X | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 5.6 | 175.616 | 176.558 | 177.504 | 178.454 | 179.406 | 180.362 | 181.321 | 182.284 | 183.250 | 184.220 |
| 5.7 | 185.193 | 186.169 | 187.149 | 188.133 | 189.119 | 190.109 | 191.103 | 192.100 | 193.101 | 194.105 |
| 5.8 | 195.112 | 196.123 | 197.137 | 198.155 | 199.177 | 200.202 | 201.230 | 202.262 | 203.297 | 204.336 |
| 5.9 | 205.379 | 206.425 | 207.475 | 208.528 | 209.585 | 210.645 | 211.709 | 212.776 | 213.847 | 214.922 |
| 6.0 | 216.00 | 217.082 | 218.167 | 219.256 | 220.349 | 221.445 | 222.545 | 223.649 | 224.756 | 225.867 |
| 6.1 | 226.981 | 228.099 | 229.221 | 230.346 | 231.476 | 232.608 | 233.745 | 234.885 | 236.029 | 237.177 |
| 6.2 | 238.328 | 239.483 | 240.642 | 241.804 | 242.971 | 244.141 | 245.314 | 246.492 | 247.673 | 248.858 |
| 6.3 | 250.047 | 251.240 | 252.436 | 253.636 | 254.840 | 256.048 | 257.259 | 258.475 | 259.694 | 260.917 |
| 6.4 | 262.144 | 263.375 | 264.609 | 265.848 | 267.090 | 268.336 | 269.586 | 270.840 | 272.098 | 273.359 |
| 6.5 | 274.625 | 275.894 | 277.168 | 278.445 | 279.726 | 281.011 | 282.300 | 283.593 | 284.890 | 286.191 |
| 6.6 | 287.496 | 288.805 | 290.118 | 291.434 | 292.755 | 294.080 | 295.408 | 296.741 | 298.078 | 299.418 |
| 6.7 | 300.763 | 302.112 | 303.464 | 304.821 | 306.182 | 307.547 | 308.916 | 310.289 | 311.666 | 313.047 |
| 6.8 | 314.432 | 315.821 | 317.215 | 318.612 | 320.014 | 321.419 | 322.829 | 324.243 | 325.661 | 327.083 |
| 6.9 | 328.509 | 329.939 | 331.374 | 332.813 | 334.255 | 335.702 | 337.154 | 338.609 | 340.068 | 341.532 |
| 7.0 | 343.00 | 344.472 | 345.948 | 347.429 | 348.914 | 350.403 | 351.896 | 353.393 | 354.895 | 356.401 |
| 7.1 | 357.911 | 359.425 | 360.944 | 362.467 | 363.994 | 365.526 | 367.062 | 368.602 | 370.146 | 371.695 |
| 7.2 | 373.248 | 374.805 | 376.367 | 377.933 | 379.503 | 381.078 | 382.657 | 384.241 | 385.828 | 387.420 |
| 7.3 | 389.017 | 390.618 | 392.223 | 393.833 | 395.447 | 397.065 | 398.688 | 400.316 | 401.947 | 403.583 |
| 7.4 | 405.224 | 406.869 | 408.518 | 410.172 | 411.831 | 413.494 | 415.161 | 416.833 | 418.509 | 420.190 |
| 7.5 | 421.875 | 423.565 | 425.259 | 426.958 | 428.661 | 430.369 | 432.081 | 433.798 | 435.520 | 437.245 |
| 7.6 | 438.976 | 440.711 | 442.451 | 444.195 | 445.944 | 447.697 | 449.455 | 451.218 | 452.985 | 454.757 |
| 7.7 | 456.533 | 458.314 | 460.100 | 461.890 | 463.685 | 465.484 | 467.289 | 469.097 | 470.911 | 472.729 |
| 7.8 | 474.552 | 476.380 | 478.212 | 480.049 | 481.890 | 483.737 | 485.588 | 487.443 | 489.304 | 491.169 |
| 7.9 | 493.039 | 494.914 | 496.793 | 498.677 | 500.566 | 502.460 | 504.358 | 506.262 | 508.170 | 510.082 |
| 8.0 | 512.00 | 513.922 | 515.850 | 517.782 | 519.718 | 521.660 | 523.607 | 525.558 | 527.514 | 529.475 |
| 8.1 | 531.441 | 533.412 | 535.387 | 537.368 | 539.353 | 541.343 | 543.338 | 545.339 | 547.343 | 549.353 |
| 8.2 | 551.368 | 553.388 | 555.412 | 557.442 | 559.476 | 561.516 | 563.560 | 565.609 | 567.664 | 569.723 |
| 8.3 | 571.787 | 573.856 | 575.930 | 578.010 | 580.094 | 582.183 | 584.277 | 586.376 | 588.480 | 590.590 |
| 8.4 | 592.704 | 594.823 | 596.948 | 599.077 | 601.212 | 603.351 | 605.496 | 607.645 | 609.800 | 611.960 |
| 8.5 | 614.125 | 616.295 | 618.470 | 620.650 | 622.836 | 625.026 | 627.222 | 629.423 | 631.629 | 633.840 |
| 8.6 | 636.056 | 638.277 | 640.504 | 642.736 | 644.973 | 647.215 | 649.462 | 651.714 | 653.972 | 656.235 |
| 8.7 | 658.503 | 660.776 | 663.055 | 665.339 | 667.628 | 669.922 | 672.221 | 674.526 | 676.836 | 679.151 |
| 8.8 | 681.472 | 683.798 | 686.129 | 688.465 | 690.807 | 693.154 | 695.506 | 697.864 | 700.227 | 702.595 |
| 8.9 | 704.969 | 707.348 | 709.732 | 712.122 | 714.517 | 716.917 | 719.323 | 721.734 | 724.151 | 726.573 |
| 9.0 | 729.00 | 731.433 | 733.871 | 736.314 | 738.763 | 741.218 | 743.677 | 746.143 | 748.613 | 751.089 |
| 9.1 | 753.571 | 756.058 | 758.551 | 761.048 | 763.552 | 766.061 | 768.575 | 771.095 | 773.621 | 776.152 |
| 9.2 | 778.688 | 781.230 | 783.777 | 786.330 | 788.889 | 791.453 | 794.023 | 796.598 | 799.179 | 801.765 |
| 9.3 | 804.357 | 806.954 | 809.558 | 812.166 | 814.781 | 817.400 | 820.026 | 822.657 | 825.294 | 827.936 |
| 9.4 | 830.584 | 833.238 | 835.897 | 838.562 | 841.232 | 843.909 | 846.591 | 849.278 | 851.971 | 854.670 |
| 9.5 | 857.375 | 860.085 | 862.801 | 865.523 | 868.251 | 870.984 | 873.723 | 876.467 | 879.218 | 881.974 |
| 9.6 | 884.736 | 887.504 | 890.277 | 893.056 | 895.841 | 898.632 | 901.429 | 904.231 | 907.039 | 909.853 |
| 9.7 | 912.673 | 915.499 | 918.330 | 921.167 | 924.010 | 926.859 | 929.714 | 932.575 | 935.441 | 938.314 |
| 9.8 | 941.192 | 944.076 | 946.966 | 949.862 | 952.764 | 955.672 | 958.585 | 961.505 | 964.430 | 967.362 |
| 9.9 | 970.299 | 973.242 | 976.191 | 979.147 | 982.108 | 985.075 | 988.048 | 991.027 | 994.012 | 997.003 |
| 10.0 | 1000.00 | 1003.003 | 1006.012 | 1009.027 | 1012.048 | 1015.075 | 1018.108 | 1021.147 | 1024.193 | 1027.244 |

$\sqrt[3]{428.661} \approx 7.54$, $\sqrt[3]{846.591} \approx 9.46$, $\sqrt[3]{213.847} \approx 5.98$